Efficient Implementation of the Divergence-Preserved ADI-FDTD Method

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***Abstract*—Unlike its conventional version, an unconditionally stable alternating-direction-implicit (ADI) finite-difference time-domain (FDTD) method was recently proposed that retains the correct divergence property but with higher computation expenditure. In this letter, a newly formulated divergence-preserved ADI-FDTD method is proposed. It takes about 41.7% less count of floating-point operations than the original divergence-preserved ADI-FDTD method without scarifying accuracy. Detailed analysis and numerical examples are presented to verify the improvement of computational efficiency.**

***Index Terms*—Alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method, computational efficiency, divergence-free, divergence preservation, floating-point operation, unconditionally stability.**

I. Introduction

ALTHOUGH the unconditionally stable alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method and its variations have been well developed for modeling electromagnetic structures, the field quantities computed have been found to be not divergence-preserved (e.g., divergence is not zero in source-free regions) [1]. To address the issue, a divergence-preserved ADI-FDTD method was proposed in [1]. Detailed formulations, stability proof, numerical dispersion, and other further analysis of the method were presented in [2]. Because of its divergence-preservation, the method has great potential in various applications, in particular in those involving charges such as electromagnetic particle-in-cell (EMPIC) simulations [3]. However, this divergence-preserved ADI-FDTD method requires 24 for-loops and 72 floating-point operations (multiplications/divisions and additions/subtractions) to update fields in one full time-step. As a result, for electrically large or high-Q structures, computational expenditure may become intolerantly and unacceptably high. Thus, an efficiency-improved divergence-preserved ADI-FDTD method is much desired.

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In this letter, based on the theory presented in [1], we propose a new efficiency-improved divergence-preserved ADI-FDTD method with reduced floating-point operation count. First, we present the general formulations of our proposed efficiency-improved divergence-preserved ADI-FDTD method. Then, we compare its computational efficiency with the original divergence-preserved ADI-FDTD method [1] and the conventional non-divergence-preserved ADI-FDTD method and its enhanced variations of [4] and [5] in terms of floating-point operation count. Numerical experiments are finally given to verify the accuracy and efficiency of the proposed method.

II. Formulations of the Proposed Efficiency-Improved Divergence-Preserved ADI-FDTD Method

Linear, lossless, isotropic, and nondispersive media with permittivity ε and permeability μ are considered. The time-dependent Maxwell's curl equations can then be written in the following matrix form:

$$\frac{\partial \mathbf{V}}{\partial t} = c(\mathbf{P}+\mathbf{M})\cdot\mathbf{V} \tag{1}$$

where

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \partial y \\ 0 & 0 & 0 & \partial z & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial x & 0 \\ 0 & \partial z & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial x & 0 & 0 & 0 \\ \partial y & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2a}$$

and

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\partial z & 0 \\ 0 & 0 & 0 & 0 & 0 & -\partial x \\ 0 & 0 & 0 & -\partial y & 0 & 0 \\ 0 & 0 & -\partial y & 0 & 0 & 0 \\ -\partial z & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial x & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2b}$$

The original divergence-preserved ADI-FDTD method [1] can be expressed as

$$\left(\mathbf{I}-\frac{c\Delta t}{2}\mathbf{P}\right)\left(\mathbf{I}+\frac{c\Delta t}{2}\mathbf{M}\right)^{-1}\mathbf{V}^{n+1} = \left(\mathbf{I}+\frac{c\Delta t}{2}\mathbf{P}\right)\left(\mathbf{I}-\frac{c\Delta t}{2}\mathbf{M}\right)^{-1}\mathbf{V}^{n}. \tag{3}$$

As shown in [2], (3) can be computed within two substeps. For the first substep

$$\left(\mathbf{I}-\frac{c\Delta t}{2}\mathbf{M}\right)\mathbf{Q}^{n+1/2} = \mathbf{V}^{n} \tag{4a}$$

$$\mathbf{V}^{n+1/2} = \left(\mathbf{I} + \frac{c\Delta t}{2}\mathbf{P}\right)\mathbf{Q}^{n+1/2}. \quad (4b)$$

For the second substep

$$\left(\mathbf{I} - \frac{c\Delta t}{2}\mathbf{P}\right)\mathbf{Q}^{n+1} = \mathbf{V}^{n+1/2} \quad (5a)$$

$$\mathbf{V}^{n+1} = \left(\mathbf{I} + \frac{c\Delta t}{2}\mathbf{M}\right)\mathbf{Q}^{n+1}. \quad (5b)$$

Here, $\mathbf{Q}$ is the intermediate variable.

Equations (4) and (5) form the original divergence-preserved ADI-FDTD method that requires more computational expenditure than other methods as shown in Table II of Section IV.

Now we substitute (4b) into (5a)

$$\left(\mathbf{I} - \frac{c\Delta t}{2}\mathbf{P}\right)\mathbf{Q}^{n+1} = \left(\mathbf{I} + \frac{c\Delta t}{2}\mathbf{P}\right)\mathbf{Q}^{n+1/2} = 2\mathbf{Q}^{n+1/2} - \left(\mathbf{I} - \frac{c\Delta t}{2}\mathbf{P}\right)\mathbf{Q}^{n+1/2}. \quad (6)$$

Then

$$\left(\frac{\mathbf{I}}{2} - \frac{c\Delta t}{4}\mathbf{P}\right)(\mathbf{Q}^{n+1} + \mathbf{Q}^{n+1/2}) = \mathbf{Q}^{n+1/2}. \quad (7)$$

Equation (7) can be reformulated with the introduction of intermediate $\mathbf{U}$ such that

$$\left(\frac{\mathbf{I}}{2} - \frac{c\Delta t}{4}\mathbf{P}\right)\mathbf{U}^{n+1/2} = \mathbf{Q}^{n+1/2} \quad (8a)$$

$$\mathbf{Q}^{n+1} = \mathbf{U}^{n+1/2} - \mathbf{Q}^{n+1/2} \quad (8b)$$

where $\mathbf{U}^{n+1/2} = \left[U_{ex}^{n+1/2}, U_{ey}^{n+1/2}, U_{ez}^{n+1/2}, U_{hx}^{n+1/2}, U_{hy}^{n+1/2}, U_{hz}^{n+1/2}\right]$.

For the second sub-time-step, we advance (4a) by one time-step and have

$$\left(\mathbf{I} - \frac{c\Delta t}{2}\mathbf{M}\right)\mathbf{Q}^{n+3/2} = \mathbf{V}^{n+1}. \quad (9)$$

By substituting (9) into (5b), we obtain

$$\left(\mathbf{I} - \frac{c\Delta t}{2}\mathbf{M}\right)\mathbf{Q}^{n+3/2} = \left(\mathbf{I} + \frac{c\Delta t}{2}\mathbf{M}\right)\mathbf{Q}^{n+1} = 2\mathbf{Q}^{n+1} - \left(\mathbf{I} - \frac{c\Delta t}{2}\mathbf{M}\right)\mathbf{Q}^{n+1} \quad (10a)$$

$$\left(\frac{\mathbf{I}}{2} - \frac{c\Delta t}{4}\mathbf{M}\right)(\mathbf{Q}^{n+3/2} + \mathbf{Q}^{n+1}) = \mathbf{Q}^{n+1}. \quad (10b)$$

Then, (8b) can be rewritten as

$$\left(\frac{\mathbf{I}}{2} - \frac{c\Delta t}{4}\mathbf{M}\right)\mathbf{U}^{n+1} = \mathbf{Q}^{n+1} \quad (11a)$$

$$\mathbf{Q}^{n+3/2} = \mathbf{U}^{n+1} - \mathbf{Q}^{n+1}. \quad (11b)$$

Equations (8) and (11) form the basic field updating equations of the proposed efficiency-improved divergence-preserved ADI-FDTD methods. Unlike (4b) and (5b) of the original divergence-preserved ADI-FDTD method, (8b) and (11b) of the proposed method involve only simple matrix subtractions and do not require matrix multiplications. Therefore, the proposed divergence-preserved method is more computationally efficient (as will be shown in detail in Section III).

Moreover, when expanded into component fields, (8) and (11) of the proposed method can be further simplified. Take the components in the x-direction of U and Q for example; for field march from the nth time-step to the $(n+1/2)$th time-step, after some mathematic manipulations, we have

$$\frac{1}{2}U_{ex}^{n+1/2} - \frac{c^2\Delta t^2}{8}\partial y \partial y U_{ex}^{n+1/2} = Q_{ex}^{n+1/2} + \frac{c\Delta t}{2}\partial y Q_{hz}^{n+1/2} \quad (12a)$$

$$Q_{ex}^{n+1} = U_{ex}^{n+1/2} - Q_{ex}^{n+1/2} \quad (12b)$$

$$Q_{hx}^{n+1} = Q_{hx}^{n+1/2} + \frac{c\Delta t}{2}\partial z U_{ey}^{n+1/2}. \quad (12c)$$

For field march from the $(n+1/2)$th time-step to the $(n+1)$th sub-time-step, we have

$$\frac{1}{2}U_{ex}^{n+1} - \frac{c^2\Delta t^2}{8}\partial z \partial z U_{ex}^{n+1} = Q_{ex}^{n+1} - \frac{c\Delta t}{2}\partial z Q_{hy}^{n+1} \quad (13a)$$

$$Q_{ex}^{n+3/2} = U_{ex}^{n+1} - Q_{ex}^{n+1} \quad (13b)$$

$$Q_{hx}^{n+3/2} = Q_{hx}^{n+1} - \frac{c\Delta t}{2}\partial y U_{ez}^{n+1}. \quad (13c)$$

As seen from the above equations, U_{hx}, U_{hy}, and U_{hz} are not required to be computed. In other words, for the $\mathbf{U}$ vector, only U_{ex}, U_{ey}, U_{ez} components needs to be updated in each time-step; as a result, memory consumption and CPU time of the proposed method are further saved.

In the field updating equations of (8) and (11)–(13), field components are not computed directly. Rather, they are related to $\mathbf{Q}$ through the following equations:

$$\left(\mathbf{I} - \frac{c\Delta t}{2}\mathbf{M}\right)\mathbf{Q}^{\frac{1}{2}} = \mathbf{V}^0 \quad (14)$$

and

$$\mathbf{V}^{n+1} = \left(\mathbf{I} + \frac{c\Delta t}{2}\mathbf{M}\right)\mathbf{Q}^{n+1}. \quad (15)$$

The efficiency of the proposed method is further discussed next.

III. Efficiency Comparison Between the Proposed Method and the Other ADI-FDTD Methods

In this section, the floating-point operation counts of the proposed divergence-preserved ADI-FDTD method are compared to those of other ADI-FDTD methods. Table I lists the counts of floating-point operations of the conventional non-divergence-preserved ADI-FDTD method and its improved variation [4], the original divergence-preserved ADI-FDTD method [1], and the proposed efficiency-improved divergence-preserved method. The numbers given in Table I are the numbers of multiplication/division or addition/subtraction operations on the right-hand sides of the field updating equations used in each method for the field to advance for a complete

TABLE I
FLOATING-POINT OPERATION COUNTS OF DIFFERENT IMPLICIT SCHEMAS WITH SECOND-ORDER CENTRAL DIFFERENCE

Scheme		Non-divergence-preserved ADI		Divergence preservation ADI	
Algorithm		Convent.	Improved	Original	Proposed
Implicit	M/D	18	6	6	6
	A/S	48	18	12	12
Explicit	M/D	12	6	18	6
	A/S	24	12	36	18
Total		102	42	72	42
For-loops		12	12	24	12

full time-step. The numbers of For-loops are for updating of all components in the x-, y-, and z-directions.

As can be seen from Table I, the count of floating-point operations of the proposed method is about 58.8% less than the conventional non-divergence-preserved ADI-FDTD method and is the same as its improved variation [4]. However, the proposed method preserves the correct divergence property. Therefore, the proposed method should be more accurate than the original ADI-FDTD method in applications such as EMPIC simulations [3] where charges are involved. In comparison to the original divergence-preserved ADI-FDTD method [1], the count is 41.7% less. It should be mentioned that in comparison to the most recently developed efficient one-step leapfrog ADI-FDTD method [5], the proposed method has a higher count of floating-point operations, but the one-step method is not divergence-preserved [6].

In terms of the memory consumption, the proposed method is similar to the conventional non-divergence-preserved ADI-FDTD method [7]. As presented in (6) and (7), $\mathbf{U}$ and $\mathbf{Q}$, which store the intermediate field values, need to be computed. More specifically, U_{ex}, U_{ey}, and U_{ez} need to be computed in every full time-step. In comparison to the one-step leapfrog ADI-FDTD method [5], the proposed method uses more memory since the one-step method does not require the computation of intermediate values. However, the proposed method is divergence-preserved, while the one-step method [5] and its conventional method [7] are not.

For the numbers of For-loops, with careful arrangements of the position of the field components, (8a) and (8b) can be combined into one For-loop; as a result, increasing the number of For-loops is avoided. That is, only half the number of the For-loops of the original divergence-preserved ADI-FDTD method [1] (i.e., half of the 24 For-loops) is computed in the proposed method.

IV. NUMERICAL EXAMPLES AND DISCUSSION

A cavity filled with air was selected [8] to verify the accuracy and efficiency of the proposed method. A grid of $250 \times 150 \times 45$ ($= 1.6875$ millions) cells with a uniform cell size of $\Delta x = \Delta y = \Delta z = 0.4$ mm was employed. The source function is $e^{-(t-\tau)^2/t_w^2}$ with $t_w = 150$ ps and $\tau = 450$ ps. A current plane source J_z was placed at $y = 75$, and the observation point was located at (125, 80, 23).

The results were obtained with the conventional FDTD method with $\mathrm{CFLN} = 1$ and the proposed method with $\mathrm{CFLN} = 1$ and 4, respectively. Fig. 1 shows the E_z obtained with the two methods, and Fig. 2 presents the absolute error of E_z computed with the proposed method in reference to the results obtained with the conventional FDTD method. The results obtained with the two methods are visibly indistinguishable. That is, the proposed efficiency-improved divergence-preserved FDTD method has good modeling accuracy.

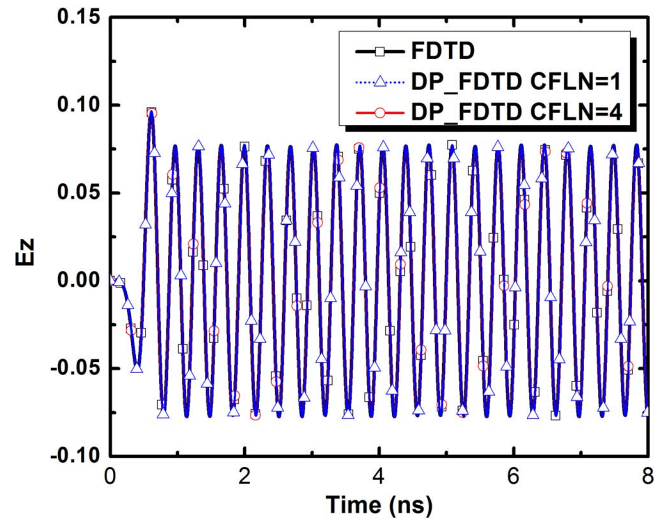

Fig. 1. E_z computed with the second-order explicit FDTD method with $\mathrm{CFLN} = 1$, and the proposed divergence preservation ADI-FDTD method with $\mathrm{CFLN} = 1$ and 4.

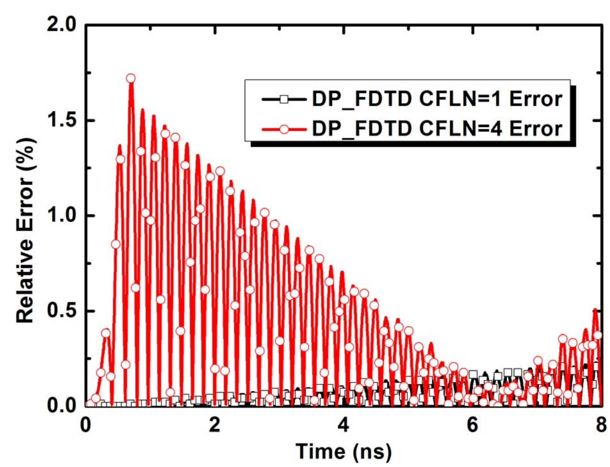

Fig. 2. Absolute relative error of E_z computed with the proposed divergence preservation ADI-FDTD method of $\mathrm{CFLN} = 1, 4$.

Table II presents resonance frequencies obtained from analytical method, the conventional FDTD method, and the proposed efficiency-improved divergence-preserved ADI-FDTD methods with $\mathrm{CFLN} = 1$ and 4. It is easy to find that the errors of the proposed method are slightly larger than the conventional FDTD method and they increase with CFLN.

Table III shows the usages of time and memory by the FDTD, the original divergence-preserved ADI-FDTD method [1], and the proposed method. Compared to the original divergence-preserved ADI-FDTD method, the proposed method took about 14% less CPU time with $\mathrm{CFLN} = 4$ and 25% less memory at the same accuracy. In comparison to the conventional FDTD method, the proposed method used 49% more memory due to

TABLE II
COMPARISON OF RESULTS WITH CONVENTIONAL FDTD AND THE PROPOSED METHOD

Analytical (GHz)	FDTD CFLN=1(GHz)	Proposed CFLN=1(GHz)	Proposed CFLN=4(GHz)
2.915	2.914	2.911	2.912
5.220	5.213	5.212	5.210
6.727	6.720	6.722	6.716
7.648	7.642	7.640	7.630

TABLE III
COMPARISON OF THE TIME AND MEMORY USED BY THE YEE'S FDTD METHOD, THE ORIGINAL DIVERGENCE-PRESERVED ADI-FDTD METHOD, AND PROPOSED METHOD

	Yee's FDTD	Original Divergence-Preserved ADI-FDTD Method		Proposed Method	
CFLN	1	1	4	1	4
Number of cells	1687500	1687500		1687500	
Number of iterations	10385	10385	2596	10385	2596
Time(s)	2206	2735	689	2396	592
Memory(Mb)	80.5	159.4	159.4	119.9	119.9

the introduction of the intermediate values of U_{ex}, U_{ey}, U_{ez}, but 73% less CPU time with $\mathrm{CFLN} = 4$.

V. CONCLUSION

In this letter, a new efficiency-improved divergence-preserved ADI-FDTD formulation has been proposed, and its efficiency comparisons to other FDTD methods are presented. It is found that the proposed method is more efficient than the original divergence-preserved ADI-FDTD method in both memory and CPU time. In comparison to the FDTD, it uses about 50% more memory but less CPU time if the time-step is chosen to be adequately large. Therefore, due to its divergence property and high efficiency, the proposed divergence-preserved ADI-FDTD method is recommended as an FDTD-based alternative technique if unconditional stability is required for modeling (e.g., highly resolved fields). Finally, it should be pointed out that further comprehensive studies on numerical properties of the proposed method, such as potential anomalous numerical propagation, as well as extension to dispersive or nonlinear media, are under way, and the results will be published in the future.

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