ECED 6400
Homework Assignments

Assignment 1 (Due: February 10, 2018)

1.1. Total internal reflection at isotropic-uniaxial medium interface

Consider an **extraordinary** plane electromagnetic wave, incident from a uniaxial crystal with dielectric constants $\epsilon_{1\perp}$ and $\epsilon_{1||}$ onto the interface $z = 0$, separating the crystal from a transparent isotropic medium with the dielectric constant ϵ_2. The optical axis of the crystal is perpendicular to the interface. Under what condition is total internal reflection possible?

1.2. Surface plasmon polariton and Brewester modes at isotropic-anisotropic medium interface

A TM-polarized **extraordinary** plane wave is incident from a uniaxial crystal with the dielectric constants $\epsilon_{1||}$ and $\epsilon_{1\perp}$ onto a planar interface $z = 0$, separating the crystal from an isotropic medium with the dielectric constant ϵ_2. The optical axis of the crystal is perpendicular to the interface.

a) Derive an expression for the complex reflection amplitude;

b) Determine the condition(s) for the Brewster mode existence and derive an expression for the Brewster angle.

c) Under what conditions can surface plasmon polaritons (SPP) propagate along the interface?

d) Determine the SPP propagation constant and skin depths in both media.

e) Determine the phase-matching condition required for SPP launch from the uniaxial crystal. What is the SPP coupling angle?

1.3. Doppler's width and Beer's length in atomic sodium vapor

1) Estimate a Doppler-induced broadening width (in Hz) of atomic sodium at room temperature.

2) Estimate the Beer absorption length in an homogeneously broadened sodium vapor with $N = 10^{14}$ cm^{-3}. You may assume that a characteristic electron displacement x_0 is of the order of Bohr's atom radius. Use the transition wavelength of 600 nm. The characteristic dipole relaxation time $T = \gamma^{-1}$ for atomic sodium is 16 nsec.

1.4. Short pulse propagation in resonant linear absorbers: Energy losses

Starting, for example, from a general expression for a pulse envelope at any propagation distance within a homogeneously broadened linear absorber, address the following questions.

(a) Consider the pulse energy, defined as

$$W(z) \propto \int_{-\infty}^{\infty} dt |\mathcal{E}(t,z)|^2,$$

Using the properties of Fourier transforms, show that the energy attenuation factor, $\Gamma(z) = W(Z)/W(0)$, is given by

$$\Gamma(z) = \frac{\int_{-\infty}^{\infty} d\nu |\tilde{\mathcal{E}}(\nu)|^2 \, \exp\left(-\frac{2\alpha z}{1+\nu^2 T^2}\right)}{\int_{-\infty}^{\infty} d\nu |\tilde{\mathcal{E}}(\nu)|^2}.$$

Here $\tilde{\mathcal{E}}(\nu)$ is a spectral amplitude of the pulse at $z = 0$, and T is a dipole relaxation time.

(b) Specify to a Gaussian pulse, $\mathcal{E}(t,0) \propto e^{-t^2/2T_p^2}$. Using the Appendix, show that for sufficiently long propagation distances, $\alpha z \gg 1$, the energy attenuation factor of an ultrashort Gaussian pulse ($T_p \ll T$) is

$$\Gamma_\infty(z) \simeq \exp\left(-\frac{2T_p}{T}\sqrt{2\alpha z}\right).$$

(c) Compare the behavior of $\Gamma_\infty(z)$ with that of a long pulse, $\Gamma_0(z) = e^{-\alpha z}$, in the long-term limit $\alpha z \gg 1$. How can you explain anomalously low energy loss rates of ultrashort pulses?

Assignment 2 (Due: March 20, 2018)

2.1. Symmetries of nonlinear optical susceptibilities in isotropic media

Use the invariance of $\chi^{(3)}_{ijkl}$ in isotropic media with respect to rotations by $\pi/4$ around the z-axis to derive the following relation among the components of $\chi^{(3)}_{ijkl}$

$$\chi^{(3)}_{xxxx} = \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyyx} + \chi^{(3)}_{xyxy}.$$

2.2 Difference-frequency generation (DFG) and optical parametric oscillator

Consider a $\chi^{(2)}$-nonlinear crystal placed inside a cavity with partially transmitting mirrors. The entering pump wave experiences parametric down-conversion to generate two waves–signal and idler–with lower frequencies. The waves bounce inside a resonator, getting amplified during each round trip. A portion of the energy leaks out of the cavity through its partially transmitting mirrors, realizing a source of powerful waves at the signal and idler frequencies. This device is known as an *optical parametric oscillator*.

a) Derive an expression for the critical intensity of a cw pump required to achieve signal gain by the oscillator in the plane-wave geometry assuming the undepleted pump approximation. The linear loss coefficient due to mirror transmission is α. You may assume perfect phase matching and neglect, for simplicity, any nonlinear losses such that $\chi_{eff}^{(2)*} = \chi_{eff}^{(2)}$. The DFG equations for this situation read

$$\frac{d\mathcal{E}_2}{dz} + \frac{\alpha}{2}\mathcal{E}_2 = \zeta_2 \mathcal{E}_3^* \tag{1}$$

and

$$\frac{d\mathcal{E}_3}{dz} + \frac{\alpha}{2}\mathcal{E}_3 = \zeta_3 \mathcal{E}_2^*, \tag{2}$$

where $\zeta_{2,3}$ are defined in the Notes.

b) Estimate the critical power for a pump beam of $\lambda \simeq 5 \times 10^{-5}$ cm, focused to a spot size of 100 μm inside an $L = 1$ cm long $LiNbO_3$ crystal with $\chi_{eff}^{(2)} \simeq 0.6 \times 10^{-11}$ m/V. You may assume all linear refractive indices $n \sim 2$ and $\alpha L \simeq 4 \times 10^{-2}$, implying a 4% power loss per a roundtrip inside the cavity.

2.3. Third harmonic generation: Beyond undepleted pump approximation

Consider the THG process for the case of plane wave geometry and perfect phase matching. In these conditions, the coupled wave equations derived in class simplify to

$$\frac{d\mathcal{E}_\omega}{dz} = \frac{3i\omega\chi_{eff}^{(3)}}{2n_\omega c}\,\mathcal{E}_{3\omega}\mathcal{E}_\omega^{*2}$$

and

$$\frac{d\mathcal{E}_{3\omega}}{dz} = \frac{3i\omega\chi_{eff}^{(3)}}{2n_{3\omega} c}\,\mathcal{E}_\omega^3$$

To simplify algebra slightly you may assume that $\chi_{eff}^{(3)}$ is real which works for lossless media.

(a) Introducing dimensionless amplitudes $\mathcal{A}_\omega$ and $\mathcal{A}_{3\omega}$ viz.,

$$\mathcal{E}_\omega = \sqrt{\frac{2I}{\epsilon_0 n_\omega c}}\mathcal{A}_\omega e^{i\phi_\omega}, \qquad \mathcal{E}_{3\omega} = \sqrt{\frac{2I}{\epsilon_0 n_{3\omega} c}}\mathcal{A}_{3\omega} e^{i\phi_{3\omega}},$$

derive the two integrals of motion,

$$\mathcal{A}_\omega^2 + \mathcal{A}_{3\omega}^2 = 1, \quad \text{(power conservation)},$$

and

$$\mathcal{A}_{3\omega}\mathcal{A}_\omega^3 \cos\theta = \Gamma,$$

where $\theta = \phi_{3\omega} - 3\phi_\omega$.

(b) Consider the particular case $\theta = \pi/2$, implying that $\Gamma = 0$. Find and sketch the dependence of the fundamental and third harmonic modes on ζ. Assume that at $\zeta = 0$ all power resides with the fundamental.

(c) Estimate the efficiency of THG process in a 1 cm long glass sample, $n_{3\omega} \simeq n_\omega \sim 1.5$ by a cw laser with $P = 1$ W. Assume that $\lambda \sim 5 \times 10^{-5}$ cm and the laser light beam is tightly focused to a size of about 10^{-2} cm. How does the efficiency change if a pulsed laser source with $P = 1$ kW is used instead?

2.4. Z-scan technique particulars

a) Using the Fourier transform technique discussed in the Lecture Notes and notations therein, derive the expression for a beam envelope in a detector aperture plane located a distance L_a away from the sample in the z-scan experiment as

$$\mathcal{E}(\rho, L_a) = \overline{\mathcal{E}}_s \sum_{m=0}^{\infty} \frac{(i\Delta\Phi_0)^m}{q_m m!} \exp\left(-\frac{\rho^2}{2q_m\sigma_m^2}\right),$$

where

$$\frac{1}{\sigma_m^2} = \frac{1+2m}{w_s^2} - \frac{ik}{R_s},$$

and

$$q_m = 1 + \frac{L_a}{R_s} + \frac{i(1+2m)L_a}{z_R}.$$

b) Using the above expression and the definition of the aperture transmittance given in the Notes, show that to the lowest nontrivial order in $\Delta\phi$, the transmittance is given by

$$T(x, \Delta\phi) \simeq 1 + \frac{4x\Delta\phi}{(x^2+9)(x^2+1)},$$

where $x = z_s/z_R$ in the notations of the Notes.

c) Assuming your translation stage allows for a 20 cm travel, how tight a focus is required such that you could capture a typical $z-$scan signature (peak-to-valley) of the transmittance curve? What is a maximum sample thickness such that the thin sample approximation, $\Delta L \ll z_R$, is satisfied for a 632 nm laser beam?

Assignment 3 (Due: April 15, 2018)

3.1. Conservation laws for co-and counter-propagating transient SRS

a) Consider transient SRS in the co-propagating geometry. The corresponding equations for dimensionless pump and Stokes pulse amplitudes, derived in the Notes, read

$$\partial_Z \overline{\mathcal{E}}_p = i\kappa\sigma\overline{\mathcal{E}}_s, \tag{3}$$

$$\partial_Z \overline{\mathcal{E}}_s = i\kappa^{-1}\sigma^*\overline{\mathcal{E}}_p. \tag{4}$$

Derive a **power** conservation law in the form

$$\kappa^{-1}|\overline{\mathcal{E}}_p|^2 + \kappa|\overline{\mathcal{E}}_s|^2 = K(T), \tag{5}$$

where the function $K(T)$ is determined by the initial conditions at the fiber input $Z = 0$: $K(T) = \kappa^{-1}|\overline{\mathcal{E}}_p(0,T)|^2 + \kappa|\overline{\mathcal{E}}_s(0,T)|^2$.

b) In the counter-propagating geometry, the governing equations can be written as

$$-\partial_Z \overline{\mathcal{E}}_p + \delta\,\partial_T \overline{\mathcal{E}}_p = i\kappa\sigma\overline{\mathcal{E}}_s, \tag{6}$$

and

$$\partial_Z \overline{\mathcal{E}}_s + \delta\,\partial_T \overline{\mathcal{E}}_s = i\kappa^{-1}\sigma^*\overline{\mathcal{E}}_p. \tag{7}$$

Show that in this case, there is an **energy** invariant in the form

$$\int_{-\infty}^{\infty} dT\,(\kappa|\overline{\mathcal{E}}_s|^2 - \kappa^{-1}|\overline{\mathcal{E}}_p|^2) = const. \tag{8}$$

Hint. You may assume that all pulses carry finite energy such that their amplitudes decay fast toward their tails.

c) Explain why the power is conserved in the co-propagating case and not in the counter-propagating one.

3.2. Self-similarity in co-propagating transient SRS

Revisit transient SRS in the co-propagating geometry. Assume that pump pulses are much shorter than the typical relaxation time of the Raman medium, $T_p \ll \gamma^{-1}$. In this coherent transient regime, relaxation effects are negligible, $\gamma \simeq 0$. Consider chirpless input pulses such that $\mathcal{E}_p^* = \mathcal{E}_p$, $\mathcal{E}_s^* = \mathcal{E}_s$.

a) Show that the SRS equations can be cast into the form

$$\partial_Z \overline{\mathcal{E}}_p = -\kappa\overline{\sigma}\overline{\mathcal{E}}_s, \tag{9}$$

$$\partial_Z \overline{\mathcal{E}}_s = \kappa^{-1} \overline{\sigma} \overline{\mathcal{E}}_p, \tag{10}$$

and

$$\partial_T \overline{\sigma} = \overline{\mathcal{E}}_p \overline{\mathcal{E}}_s, \tag{11}$$

where $\sigma = i\overline{\sigma}$, $\overline{\sigma}^* = \overline{\sigma}$.

Show that the power conservation law (5) implies the following parametrization of the pulse amplitudes

$$\overline{\mathcal{E}}_p = (\kappa K)^{1/2} \cos\theta/2, \tag{12}$$

and

$$\overline{\mathcal{E}}_s = (K/\kappa)^{1/2} \sin\theta/2. \tag{13}$$

Here θ is a real phase. Derive the so-called sin-Gordon equation for the phase,

$$\partial^2_{Z\overline{T}} \theta = \sin\theta, \tag{14}$$

where the effective time $\overline{T}$ is defined as

$$\overline{T} = \int_{-\infty}^{T} ds\, K(s). \tag{15}$$

b) Show that the sin-Gordon equation admits a **self-similar** solution such that the phase depends only on a combination of time and propagation distance variables as

$$\theta = \theta(\eta), \qquad \eta = 2\sqrt{Z\overline{T}}.$$

Show that the phase dynamics is then governed by the following ordinary differential equation

$$\theta''_{\eta\eta} + \tfrac{1}{\eta}\theta'_{\eta} = \sin\theta. \tag{16}$$

This self-similarity is **universal** in the sense that any initial pulse profile asymptotically evolves toward the same self-similar shape in the long-term limit.

c) So far we have ignored fiber losses. Assuming the transient regime, $T_p \ll \gamma^{-1}$, and dispersion-flat fibers, the linear loss coefficient α has to be frequency-independent, implying that the governing SRS equations read

$$\partial_Z \overline{\mathcal{E}}_p + \tfrac{\alpha}{2}\mathcal{E}_p = -\kappa \overline{\sigma} \overline{\mathcal{E}}_s,$$

$$\partial_Z \overline{\mathcal{E}}_s + \tfrac{\alpha}{2}\mathcal{E}_s = \kappa^{-1} \overline{\sigma} \overline{\mathcal{E}}_p,$$

and

$$\partial_T \overline{\sigma} = \overline{\mathcal{E}}_p \overline{\mathcal{E}}_s.$$

Transforming to the scaled fields,

$$\overline{\mathcal{E}}_p = \tilde{\mathcal{E}}_p e^{-\alpha Z/2}, \qquad \overline{\mathcal{E}}_s = \tilde{\mathcal{E}}_s e^{-\alpha Z/2};$$

show that we can still introduce the phase θ the same way as before and it obeys the sine-Gordon equation as

$$\partial^2_{\overline{ZT}}\theta = \sin\theta,$$

in the scaled variables,

$$\overline{T} = \int_{-\infty}^{T} ds\, K(s), \qquad \overline{Z} = \tfrac{1}{\alpha}\left(1 - e^{-\alpha Z}\right).$$

Thus, fiber losses do not destroy the SRS self-similarity. Rather, they reduce the effective interaction length from the actual fiber length L to $L_{\text{eff}} = (1 - e^{-\alpha L})/\alpha$.

3.3. Self-similar pump and Stokes pulse profiles

Solve Eq. (16) numerically subject to the initial conditions $\theta(0) = 2\pi$ and $\theta'_\eta(0) = 0$, corresponding to the case of a strong pump pulse entering a Raman medium in its ground state. Plot the pump and Stokes profiles in one figure as functions of the similarity variable.

3.4. SRS generation threshold in hollow-core photonic crystal fibers

Estimate the critical energy required to initiate the SRS process from noise with 10 ns pump pulses at $\lambda_p = 1.06\mu$m in a 1 m long lossless hollow-core photonic crystal fiber filled with molecular hydrogen. The effective core area of the fiber is $30\mu\text{m}^2$. You may assume that $G_{th} = 25$.

Appendix

1. Gaussian integrals

You may find useful the following Gaussian integrals

$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{17}$$

and

$$\int_{-\infty}^{+\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right). \tag{18}$$

2. Laplace method for asymptotic evaluation of integrals

Consider an integral

$$I = \int_{-\infty}^{+\infty} dx e^{-\lambda f(x)},$$

for an arbitrary **real** function $f(x)$ in the limit of very large λ ($\lambda \to +\infty$). The main contribution to the integral comes from the neighborhood of the point at which f attains minimum. Let us call such a point x_0, and expand f in the vicinity of x_0 in a Taylor series up to the second order:

$$f(x) \simeq f(x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2.$$

Here at x_0

$$f'(x_0) = 0, \qquad f''(x_0) > 0.$$

The integral I is then

$$I \simeq e^{-\lambda f(x_0)} \int_{-\infty}^{+\infty} dx e^{-\lambda f''(x_0)(x-x_0)^2/2}.$$

Introducing the variable $s = x - x_0$, we can rewrite the integral as

$$I \simeq e^{-\lambda f(x_0)} \int_{-\infty}^{+\infty} ds e^{-\lambda f''(x_0)s^2/2}.$$

The integral on the r.h.s. can be evaluated using Eq. (17). The result is

$$I \simeq \sqrt{\frac{2\pi}{\lambda f''(x_0)}}\, e^{-\lambda f(x_0)}. \tag{19}$$

In case of multiple minima x_k, Eq. (19) is naturally generalized to

$$I \simeq \sum_k \sqrt{\frac{2\pi}{\lambda f''(x_k)}}\, e^{-\lambda f(x_k)}. \tag{20}$$