Solutions to Final 2017

Problem 1

According to Maxwell's equation,

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E} = -\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial_x & \partial_y & \partial_z \\ A\, y \cos\omega t & -A\, x \cos\omega t & 0 \end{vmatrix} = 2A\, \cos\omega t\, \mathbf{a}_z$$

Problem 2

a) Placing the origin of our spherical coordinates at the position of the charge Q, we infer from Gauss's law that

$$D \times 4\pi r^2 = Q,$$

implying that the electric flux density **anywhere** is given by

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r.$$

Thus, the electric field,

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon_1 r^2}\mathbf{a}_r, & r < a; \\ \frac{Q}{4\pi\epsilon_2 r^2}\mathbf{a}_r, & a < r < b; \\ \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r, & r > b. \end{cases}$$

b) The amount of energy stored in the shaded region is

$$W = \tfrac{1}{2}\epsilon_2 \int dv\, E^2 = \tfrac{1}{2}\epsilon_2 \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta}_{=4\pi} \int_a^b dr r^2 \tfrac{Q^2}{16\pi^2\epsilon_2^2 r^4} = \tfrac{Q^2}{8\pi\epsilon_2}\left(\tfrac{1}{a} - \tfrac{1}{b}\right).$$

c) It then follows at once that $\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E}$. Thus,

$$\mathbf{P} = \begin{cases} \frac{Q}{4\pi r^2}\left(1 - \epsilon_0/\epsilon_1\right)\mathbf{a}_r, & r < a; \\ \frac{Q}{4\pi r^2}\left(1 - \epsilon_0/\epsilon_2\right)\mathbf{a}_r, & a < r < b; \\ 0, & r > b. \end{cases}$$

By definition,

$$\rho_{pv} = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\partial_r(r^2 P_r) = -\frac{1}{r^2}\partial_r\left[r^2 \frac{Q}{4\pi r^2}\left(1 - \epsilon_0/\epsilon_j\right)\right] = 0,$$

where $j = 1, 2$. The volume polarization charge density in free space is, of course, zero because $P = 0$ there. Thus $\rho_{pv} = 0$ **everywhere**.

d) The field outside the concentric spheres is generated by the only net charge in the problem, the charge Q at the origin. By definition,

$$W = q(V_f - V_i),$$

where $V_i = 0$, the potential far away from the point charge at the center and $V_f = \frac{Q}{4\pi\epsilon_0 b}$ is a potential at the surface of the outer sphere. Thus,

$$W = \frac{Qq}{4\pi\epsilon_0 b}.$$

Problem 3

a) The magnetic flux density must be solenoidal, $\nabla \cdot \mathbf{B} = 0$ (no magnetic charges). A simple check confirms that

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\partial_\rho\left(\rho \frac{1}{\rho} e^{-\rho^2}\right)\cos\phi + 2\rho\, e^{-\rho^2}\frac{1}{\rho}\partial_\phi(\sin\phi) = -2\, e^{-\rho^2}\cos\phi + 2\, e^{-\rho^2}\cos\phi = 0.$$

b) The flux is equal to zero because the field is solenoidal; in other words, using Gauss's theorem

$$\oint d\mathbf{S}\cdot\mathbf{F} = \int dv \nabla\cdot\mathbf{F} = 0.$$

Alternatively, the flux through the top and bottom cylinder surfaces is zero because $\mathbf{a}_n = \pm\mathbf{a}_z$ and $\mathbf{a}_z \cdot \mathbf{a}_\rho = \mathbf{a}_z \cdot \mathbf{a}_\phi = 0$. The flux through the walls with $d\mathbf{S} = \mathbf{a}_\rho dz R d\phi$ is given by

$$\oint d\mathbf{S}\cdot\mathbf{F} = \int_0^H dz \frac{1}{R} e^{-R^2} \underbrace{\int_0^{2\pi} d\phi \cos\phi}_{=0} = 0.$$

Problem 4

a) Ampère's law implies that

$$\mathbf{J} = \nabla\times\mathbf{H} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ 2\cos\theta/r & -\sin\theta & 0 \end{vmatrix} = \frac{2\sin\theta}{r^2}\mathbf{a}_\phi.$$

b) The continuity equation states that

$$\partial_t\rho_v = -\nabla\cdot\mathbf{J} = -\frac{1}{r^2\sin\theta}\partial_\phi J_\phi = -\frac{1}{r^2\sin\theta}\partial_\phi\left(\frac{2\sin\theta}{r^2}\right) = 0.$$

c) By definition,

$$I = \int d\mathbf{S}\cdot\mathbf{J} = 0,$$

because $d\mathbf{S} = \mathbf{a}_r R d\theta R\sin\theta d\phi$ and $d\mathbf{S}\cdot\mathbf{J} = 0$ since $\mathbf{a}_r\cdot\mathbf{a}_\phi = 0$.

Problem 5

a) Assume from the charge distribution symmetry that $V(x, y, z) = V(z)$. The potential distribution then follows from Poisson's equation written in the Cartesian coordinates as

$$\frac{d^2V}{dz^2} = -(\rho_0/\epsilon)e^{-z/a}.$$

Integrating twice, yields

$$V(z) = C_2 + C_1 z - a^2(\rho_0/\epsilon)e^{-z/a}.$$

Imposing the boundary conditions, $V \to 0$ as $z \to \infty$ implies $C_1 = C_2 = 0$. Thus,

$$V(z) = -a^2(\rho_0/\epsilon)e^{-z/a}.$$

b) The electric field is $\mathbf{E} = -\nabla V$. Hence,

$$\mathbf{E} = -\frac{\partial V}{\partial z}\mathbf{a}_z = -a(\rho_0/\epsilon)e^{-z/a}\mathbf{a}_z,$$

in the dielectric. In free space, $z < 0$, the Laplace equation reads,

$$\frac{d^2V}{dz^2} = 0,$$

immediately implying the linear solution,

$$V(z) = Az + B,$$

where A and B are unknown constants. The electric field in the lower half-space is uniform

$$\mathbf{E} = -\nabla V = -A\mathbf{a}_z$$

To determine its magnitude A, we stipulate that the normal component of $\mathbf{D}$ be continuous across the interface $z = 0$:

$$\epsilon \mathbf{E}_>|_{z=0} \cdot \mathbf{a}_z = \epsilon_0 \mathbf{E}_<|_{z=0} \cdot \mathbf{a}_z \Longrightarrow a\rho_0 = \epsilon_0 A,$$

implying that

$$A = a\rho_0/\epsilon_0.$$

The final answer is then

$$\mathbf{E} = \begin{cases} -a(\rho_0/\epsilon)e^{-z/a}\mathbf{a}_z, & z > 0, \\ -a(\rho_0/\epsilon_0)\mathbf{a}_z, & z < 0. \end{cases}$$