Solutions to Final 2018

Problem 1

The system is equivalent to two parallel-plate capacitors of capacitances $C_1 = \epsilon A/h$ and $C_2 = \epsilon_0 A/(d-h)$ connected in a series. Hence,

(a)

$$C^{-1} = C_1^{-1} + C_2^{-1} = A^{-1}[h\epsilon^{-1} + (d-h)\epsilon_0^{-1}], \implies C = \frac{\epsilon_0 \epsilon A}{\epsilon_0 h + \epsilon(d-h)}.$$

(b)

$$W_E = \frac{CV_0^2}{2} = \frac{\epsilon_0 \epsilon A V_0^2}{2[\epsilon_0 h + \epsilon(d-h)]}.$$

Problem 2

According to Faraday's law,

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

The magnetic flux,

$$\Phi = \int d\mathbf{S} \cdot \mathbf{B}, \qquad d\mathbf{S} = \mathbf{a}_z \rho d\phi d\rho;$$

and the magnetic flux density can be expressed in the cylindrical coordinates as

$$\mathbf{B} = \mathbf{a}_z B_0 \rho \cos\phi \cos\omega t.$$

It follows that

$$\Phi = B_0 \cos\omega t \int_0^R d\rho\, \rho^2 \int_0^\alpha d\phi\, \cos\phi (\mathbf{a}_z \cdot \mathbf{a}_z) = B_0 \cos\omega t \sin\alpha R^3/3.$$

Finally,

$$\mathcal{E} = \left(\frac{\omega B_0 R^3}{3}\right) \sin\alpha \sin\omega t.$$

Problem 3

a) By definition,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta \mathbf{a}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \frac{1}{2} B_0 r^2 \sin^2\theta \end{vmatrix} = B_0(\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta) = B_0 \mathbf{a}_z.$$

By the same token,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \frac{B_0 a^3 \sin^2\theta}{2r} \end{vmatrix} = \frac{B_0 a^3}{2r^3}(2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta).$$

b) By definition in free space, $\mathbf{B} = \mu_0 \mathbf{H}$. Further,

$$W_m = \tfrac{1}{2}\int dv\,(\mathbf{B}\cdot\mathbf{H}) = \tfrac{1}{2\mu_0}\int dv\,(\mathbf{B}\cdot\mathbf{B})$$

Thus, inside a sphere of radius a,

$$\mathbf{B}\cdot\mathbf{B} = B_0^2(\cos^2\theta + \sin^2\theta) = B_0^2.$$

It follows that

$$W_m = \frac{B_0^2}{2\mu_0} v_{\text{sphere}} = \frac{2\pi B_0^2 a^3}{3\mu_0}.$$

Problem 4

We employ Bio-Savart's law,

$$d\mathbf{H}_P = \frac{Id\mathbf{l}\times\mathbf{R}}{4\pi R^3}.$$

1) Segments "1" and "3" are straight, implying that $Id\mathbf{l} = Idx\mathbf{a}_x$ and $\mathbf{R} = \pm x\mathbf{a}_x$. It follows at once that $Id\mathbf{l}\times\mathbf{R} = 0$ and these segments don't contribute to the field at P.
2) On segment "2", we have, $Id\mathbf{l} = -Ibd\phi\mathbf{a}_\phi$, and $\mathbf{R} = -b\mathbf{a}_\rho$; the minus signs come from the fact that the positive direction of $\mathbf{a}_\phi$ is counterclockwise and $\mathbf{R}$ is a distance from a current element to the observation point. Putting it all together,

$$d\mathbf{H}_P = \frac{Ibd\phi\mathbf{a}_\phi \times b\mathbf{a}_\rho}{4\pi b^3},$$

Observing that $\mathbf{a}_\phi \times \mathbf{a}_\rho = -\mathbf{a}_z$, we obtain,

$$\mathbf{H}_P = -\frac{I}{4\pi b}\mathbf{a}_z \int_0^\pi d\phi = -\frac{I}{4b}\,\mathbf{a}_z.$$

Problem 5

a) In the Poisson equation, as the charge density depends only on z, so does the potential, $V = V(z)$. Hence,

$$\frac{d^2V}{dz^2} = -(\rho_0/\epsilon)\,e^{-\kappa|z|}.$$

Consider the upper half-space first, $z > 0$, $\epsilon = \epsilon_>$. We have

$$\frac{d^2V_>}{dz^2} = -(\rho_0/\epsilon_>)e^{-\kappa z}.$$

Integrating twice, we obtain,

$$V_>(z) = C_1 z + C_2 - \frac{\rho_0}{\kappa^2\epsilon_>}e^{-\kappa z}.$$

First, the potential has to be finite at any z, implying that $C_1 = 0$. Second, far away, $z \to +\infty$, the potential tends to a zero reference potential resulting in $C_2 = 0$. Thus,

$$V_>(z) = -\frac{\rho_0}{\kappa^2\epsilon_>}e^{-\kappa z}.$$

By the same token,

$$V_<(z) = -\frac{\rho_0}{\kappa^2\epsilon_<}e^{\kappa z}.$$

b) By definition,

$$\mathbf{E} = -\nabla V = -\mathbf{a}_z dV/dz.$$

It follows that

$$\mathbf{E} = \begin{cases} -\left(\frac{\rho_0}{\kappa\epsilon_>}\right) e^{-\kappa z}\,\mathbf{a}_z, & z > 0; \\ \left(\frac{\rho_0}{\kappa\epsilon_<}\right) e^{\kappa z}\,\mathbf{a}_z, & z < 0. \end{cases}$$

c) It follows from the boundary conditions that

$$\rho_s = \mathbf{a}_{n21} \cdot (\mathbf{D}_1 - \mathbf{D}_2)|_{z=0}.$$

In our case, $\mathbf{a}_{n21} = \mathbf{a}_z$, $\mathbf{D}_2 = \epsilon_< \mathbf{E}_< = (\rho_0/\kappa)e^{\kappa z}\mathbf{a}_z$, and $\mathbf{D}_1 = \epsilon_> \mathbf{E}_> = -(\rho_0/\kappa)e^{\kappa z}\mathbf{a}_z$, implying that

$$\rho_s = -2\rho_0/\kappa.$$