Electromagnetic Waves & Propagation
ECED4301

Lecture Notes

Chapter 1

Vector Algebra: A Brief Review

1.1 Scalars and vectors

In general, there are two kinds of objects one deals with in vector algebra: **scalars and vectors**. While the former have only magnitude, the latter are characterized by their **magnitudes and directions**. Physical quantities such as mass, density, temperature, and charge, say, are scalars, whereas a velocity, or a force is a vector. A unit vector–which has a unit magnitude–can always be formed by dividing a vector by its magnitude. For instance,

$$\mathbf{a} = \frac{\mathbf{A}}{|\mathbf{A}|},$$

is a unit vector directed along $\mathbf{A}$. A vector $\mathbf{A}$ can be geometrically represented as an arrow; the length of the arrow equals the magnitude of $\mathbf{A}$, and the arrow points into the direction of $\mathbf{A}$ as is seen in Fig. 1.1.

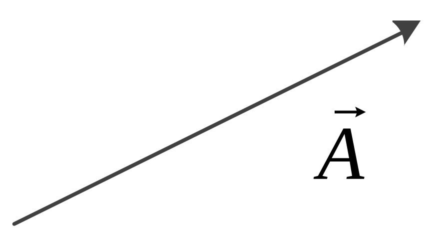

Figure 1.1: Geometric representation of a vector $\mathbf{A}$.

Another way to represent a vector is through its three–in a three-dimensional space, of course–components in a suitably chosen coordinate system. In this course, we will be exclusively working with the Cartesian coordinates such that any vector $\mathbf{A}$ can be represented in terms of its three coordinates (A_x, A_y, A_z), or alternatively,

$$\boxed{\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z}.$$

Here $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ are three mutually orthogonal unit vectors, (see Fig.1.2). The vector magnitude can be determined using the Pythagoras's theorem,

$$\boxed{|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}}.$$

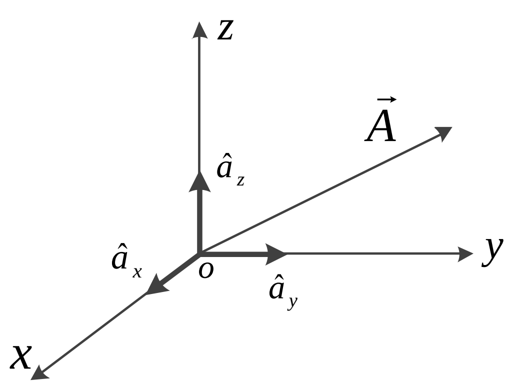

Figure 1.2: Decomposition of a vector $\mathbf{A}$ in the Cartesian coordinate system.

1.2 Vector addition and subtraction

Two vectors $\mathbf{A}$ and $\mathbf{B}$ can be added and/or subtracted component by component,

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z.$$

Geometrically, the vector addition can be represented using either a parallelogram rule or a head-to-tail rule as depicted in Fig. 1.3. The subtraction is inverse to addition. As follows from the definition, the vector addition/ subtraction obeys commutativity and associativity properties, implying that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \qquad \text{(commutativity)},$$

and

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}, \qquad \text{(associativity)}.$$

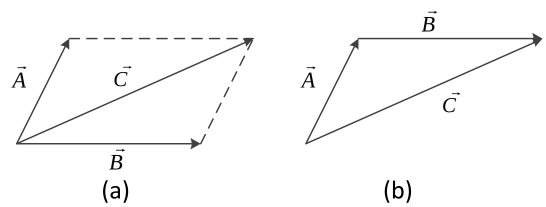

Figure 1.3: Illustrating vector addition: (a) parallelogram rule and (b) head-to-tail rule.

Also a vector can be multiplied by a scalar, implying each vector component is multiplied by a scalar,

$$k\mathbf{A} = kA_x\mathbf{a}_x + kA_y\mathbf{a}_y + kA_z\mathbf{a}_z.$$

A product of a scalar and a vector sum/difference obeys the distributive law,

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}.$$

1.3 Vector multiplication

There are two kinds of vector products: dot or scalar product and cross or vector product.

Definition. The **dot product** of two vectors $\mathbf{A}$ and $\mathbf{B}$, written as $\mathbf{A} \cdot \mathbf{B}$ is defined as a product of the vector magnitudes times the cosine of the smaller angle between them when the two are drawn tail,

$$\boxed{\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta_{AB}}. \tag{1.1}$$

Implication. As follows from the definition, the two vectors are orthogonal iff their scalar product is equal to zero.

In terms of the vector coordinates,

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1.2}$$

Note that the dot product always results in a scalar quantity. The dot product obeys the commutative and distributive rules

- $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$, commutativity;
- $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$, distributivity.

As a corollary of the definition,

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2,$$

implying an alternative way of determining the vector magnitude without resorting to vector components,

$$\boxed{|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}}.$$

Further, the mutual orthogonality of the Cartesian unit vectors implies that

$$\boxed{\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_x \cdot \mathbf{a}_z = 0}, \tag{1.3}$$

and

$$\boxed{\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1}. \tag{1.4}$$

Example. 1.1. Show that $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = |\mathbf{A}|^2 - |\mathbf{B}|^2$.
Solution. Using the properties of the dot product: $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{B} = |\mathbf{A}|^2 - |\mathbf{B}|^2$.

Definition. The **cross product** of two vectors $\mathbf{A}$ and $\mathbf{B}$, written as $\mathbf{A} \times \mathbf{B}$ is a vector whose magnitude is the area of the parallelogram formed by $\mathbf{A}$ and $\mathbf{B}$–see Fig.1.4–and is in the direction determined by the right-handed cork screw rule illustrated in Fig.1.5.
It follows that

$$\boxed{\mathbf{A} \times \mathbf{B} = AB \sin\theta_{AB} \mathbf{a}_n},$$

where $\mathbf{a}_n$ is a unit normal to the plane containing $\mathbf{A}$ and $\mathbf{B}$.

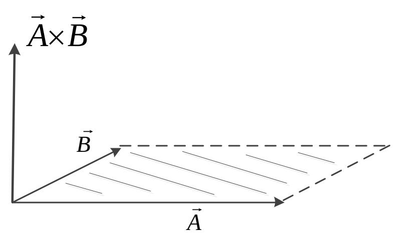

Figure 1.4: Illustrating the cross-product.

In the coordinate representation,

$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z. \end{vmatrix}}$$

The cross product obeys the following rules

- $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$;
- $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$;

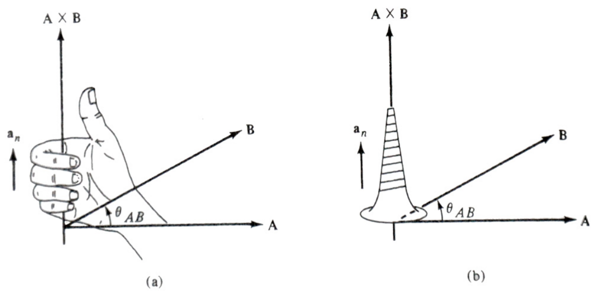

Figure 1.5: The right-handed cork-screw rule.

- $\mathbf{A} \times \mathbf{A} = 0$.

Also, the mutual cross products of the Cartesian unit vectors obey the rule

$$\boxed{\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z}, \tag{1.5}$$

with cyclic permutations for the right-handed Cartesian system as is shown in Fig. 1.6.

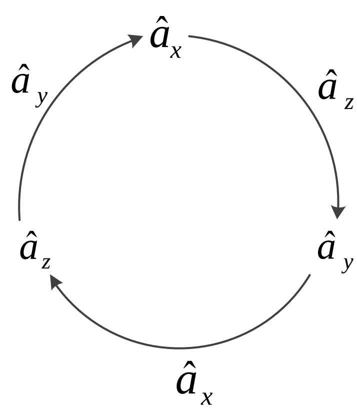

Figure 1.6: Illustrating unit vector cross products under cyclic permutations.

Example. 1.2. Show that $(\mathbf{A}+\mathbf{B}) \times (\mathbf{A}-\mathbf{B}) = 2\mathbf{B} \times \mathbf{A}$.
Solution. Using the properties of the cross product: $(\mathbf{A}+\mathbf{B}) \times (\mathbf{A}-\mathbf{B}) = \mathbf{A} \times \mathbf{A} + \mathbf{B} \times \mathbf{A} - \mathbf{A} \times \mathbf{B} - \mathbf{B} \times \mathbf{B} = 2\mathbf{B} \times \mathbf{A}$.
Example. 1.3. Given, $\mathbf{a}_x \times \mathbf{A} = -\mathbf{a}_y + 2\mathbf{a}_z$ and $\mathbf{a}_y \times \mathbf{A} = \mathbf{a}_x - 2\mathbf{a}_z$, Find A.
Solution. Assume that $\mathbf{A} = a\mathbf{a}_x + b\mathbf{a}_y + c\mathbf{a}_z$. *It follows that* $\mathbf{a}_x \times \mathbf{A} = b(\mathbf{a}_x \times \mathbf{a}_y) + c(\mathbf{a}_x \times \mathbf{a}_z) = b\mathbf{a}_z - c\mathbf{a}_y = -\mathbf{a}_y + 2\mathbf{a}_z$. *Hence,* $c = 1$ *and* $b = 2$. *Similarly,* $\mathbf{a}_y \times \mathbf{A} = -a\mathbf{a}_z + c\mathbf{a}_x = \mathbf{a}_x - 2\mathbf{a}_z$, *implying that* $c = 1$ *and* $a = 2$. *Thus* $\underline{\mathbf{A} = 2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}$.

As a consequence of the scalar and cross product definitions, we can infer that the scalar triple product can be represented as

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})}. \tag{1.6}$$

In the Cartesian coordinates, the scalar triple product can be written as

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}} \tag{1.7}$$

Finally, the vector triple product can be expressed as using "bac-cab" mnemonic rule in the form

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \tag{1.8}$$

Example 1. 4. Show that $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ is a volume of a parallelepiped having $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ as three contiguous edges.
Solution. $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \underbrace{|\mathbf{A}|\cos\theta}_{\text{height}}\underbrace{|\mathbf{B}\times\mathbf{C}|}_{\text{area}}$*, see the sketch below.*

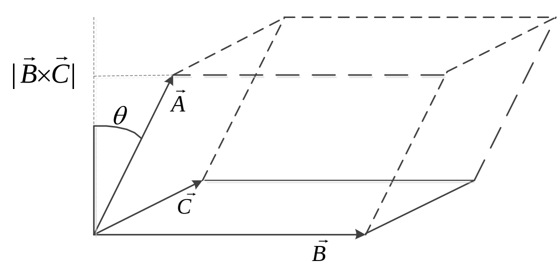

Figure 1.7: Geometric illustration of the scalar triple product.

Example 1.5. Given $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{C}$ and $\mathbf{A}\times\mathbf{B} = \mathbf{A}\times\mathbf{C}$, and $\mathbf{A}$ is not a null vector, show that $\mathbf{B} = \mathbf{C}$.
Solution. Choose the x-axis along the direction of $\mathbf{A}$. It follows that $\mathbf{A} = A\mathbf{a}_x$ where $A \neq 0$. Assume further that $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$ and $\mathbf{C} = C_x\mathbf{a}_x + C_y\mathbf{a}_y + C_z\mathbf{a}_z$. $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{C}$ then implies that $B_x = C_x$, and $\mathbf{A}\times\mathbf{B} = \mathbf{A}\times\mathbf{C}$ implies that $B_y = C_y$ as well as $B_z = C_z$. As the components are the same, the vectors are equal.

1.4 Complex numbers and phasors

Definition. A **complex number** z can be expressed in the so-called rectangular form as

$$\boxed{z = u + jv},$$

where $j = \sqrt{-1}$ and u and v are real numbers. Alternatively, it can be expressed in the polar form as

$$\boxed{z = re^{j\phi} = r(\cos\phi + j\sin\phi)},$$

where the magnitude r and phase ϕ can be written as

$$\boxed{r = \sqrt{u^2 + v^2}, \qquad \phi = \tan^{-1} v/u}.$$

Geometrically, z can be represented as a ray in the uv plane making the angle ϕ with the u-axis, see Fig. 1.8.

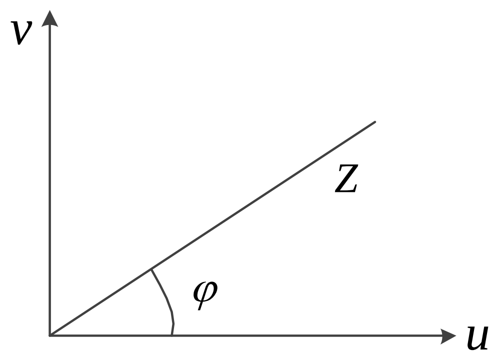

Figure 1.8: Polar form of a complex number.

Given two complex numbers, $z_1 = u_1 + jv_1 = r_1 e^{j\phi_1}$ and $z_2 = u_2 + jv_2 = r_2 e^{j\phi_2}$, the result of their addition or subtraction can be most easily expressed in the rectangular form:

$$z_1 \pm z_2 = u_1 \pm u_2 + j(v_1 \pm v_2).$$

On the other hand, their multiplication and division are more naturally expressed in the polar form as

$$z_1 z_2 = r_1 r_2 e^{j(\phi_1 + \phi_2)}, \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\phi_1 - \phi_2)}.$$

One can also introduce complex conjugation by the definition

$$\boxed{z^* = u - jv = re^{-j\phi}}.$$

In the polar form, a complex number is not uniquely defined such that

$$z = re^{j\phi} e^{j2\pi k}, \quad k = 0, \pm 1, \pm 2, \pm 3 \ldots.$$

This is because $e^{j2\pi k} = 1$ for any integer k. The latter form comes in handy whenever we want to find roots of a complex number. In general all nth roots can be represented as

$$z^{1/n} = r^{1/n} e^{j\phi/n} e^{j2\pi k/n}.$$

For example, if $n = 2$, there are only two distinct roots corresponding to $k = 0$ and $k = 1$; since $e^{j0} = 1$ and $e^{j\pi} = \cos\pi + j\sin\pi = -1$, we obtain

$$\sqrt{z} = \pm\sqrt{r} e^{j\phi/2}.$$

Definition. A **time-harmonic signal** varies sinusoidally with time.
Definition. A **phasor** represents a complex signal with a time-harmonic phase.
Thus any physical time-harmonic signal $\psi(t) = a\cos(\omega t + \theta)$, where ω and θ are constant frequency and initial phase, respectively, can be represented in terms of a complex phasor $\psi_0 e^{j\omega t}$ as

$$\boxed{\psi(t) = \text{Re}(\psi_0 e^{-j\omega t})}.$$

Here Re denotes the real part of the complex signal and the complex amplitude ψ_0 can be represented as

$$\boxed{\psi_0 = a\, e^{j\theta}},$$

where a is a real amplitude. The generalization to the phasor form of a vector time-harmonic signal is straightforward:

$$\mathbf{A}(t) = \text{Re}(\mathbf{A}_0 e^{-j\omega t}), \qquad \mathbf{A}_0 = |\mathbf{A}_0| e^{j\theta}.$$

Example 1. 6. The complex impedance of a monochromatic electromagnetic wave of frequency ω, propagating in a lossy medium is defined as

$$\eta = \sqrt{\frac{\mu/\epsilon}{1 + \frac{j\sigma}{\epsilon\omega}}}.$$

Here μ, η and σ are constitutive parameters of the medium. Express η in the polar form.
Solution. Multiplying the numerator and denominator inside the square root by $(1 - j\sigma/\epsilon\omega)$, we obtain

$$\eta = \frac{\sqrt{\mu/\epsilon}\,\left(1 - \frac{j\sigma}{\epsilon\omega}\right)^{1/2}}{\left[1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/2}} = \frac{\sqrt{\mu/\epsilon}\, e^{j\theta_\eta}}{\left[1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4}} = |\eta| e^{j\theta_\eta},$$

where

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\epsilon\omega}.$$

Chapter 2

Electromagnetic Fields and Maxwell's Equations

2.1 Charges, currents and electromagnetic fields

Definition. An **electric charge** Q quantifies the capacity of an object for electromagnetic interaction–the greater the charge the stronger the interaction. The charges could be positive or negative; the charges of the opposite signs attract to each other while those of the same sign repel from one another.
The interaction force between the two point charges Q_1 and Q_2, separated a distance R_{12} is determined by the Coulomb law

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2 \mathbf{R}_{12}}{R_{12}^3}}, \tag{2.1}$$

where $\mathbf{R}_{12}$ is a radius vector from charge Q_1 to Q_2 and ϵ_0 is the so-called **free space permittivity**, given in the SI units by the expression

$$\boxed{\epsilon_0 = \frac{10^{-9}}{36\pi}}, \text{ F/m} \tag{2.2}$$

Definition. An **electric current** is a flow of electric charges past a point or within a conductor. The current I is a time rate of change of the charge Q,

$$\boxed{I = \frac{dQ}{dt}}. \tag{2.3}$$

The charge are measured in **coulombs**, C and the currents are measured in **ampères**, abbreviated A. The smallest charge encountered in nature is the electron charge e, which is equal to -1.60219×10^{-19} C.

The electric charges and currents (moving charges) are the sources of electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields, respectively. The vector field $\mathbf{E}$ is known as the **electric field**

intensity or electric field strength and it is measured in volts per square meter, V/m^2. The field $\mathbf{B}$ is more precisely referred to as the **magnetic flux density** for the reasons that become clear shortly and it is measured in webers per square meter, Wb/m^2. To help visualize the behavior of electric and magnetic fields in space, we introduce the concept of electric and magnetic field lines.

Definition. **The electric field lines** are, in general, curves in space such that at any given point on the line, the electric field is **tangential** to the line. As electric charges are sources/sinks of the field, the electric field lines start at positive (source) and end at the negative (sink) charges. Alternatively, if the electric field is generated by a time-dependent magnetic field, its lines can be closed. These possibilities are illustrated in Fig. 2. 1.

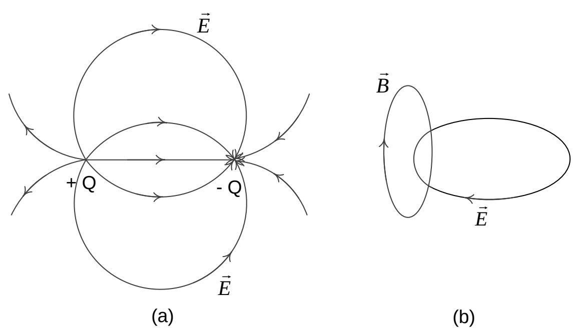

Figure 2.1: Lines of the electric field generated by (a) static electric charges and (b) a time-dependent magnetic field.

Definition. **The magnetic flux density lines** are, in general, continuous curves in space such that at any given point on the line, the magnetic flux density is **tangential** to the line. As no static magnetic charges have so far been discovered in the nature, there are no static sources of magnetic fields–the latter are generated by moving electric charges. Therefore, the magnetic flux density lines are either **closed** or go to infinity. A natural question then arises regarding the quantitative description of electric and magnetic fields: How can one quantify and measure $\mathbf{E}$ and $\mathbf{B}$ at a given point in space?

To answer this question, let us consider a small test charge q at rest. It is known from the experiment that the charge placed in an electric field $\mathbf{E}$ generated by some other charges experiences the force

$$\boxed{\mathbf{F}_e = q\mathbf{E}}. \tag{2.4}$$

It follows at once from Eq. (2.4) that the electric field at a position of the test charge is simply the force per unit charge and it can be determined as

$$\boxed{\mathbf{E} = \frac{\mathbf{F}_e}{q}}. \tag{2.5}$$

Note that the definition (2.5) is unambiguous, provided the test charge is so small that it does not alter the field at its location. Thus, the strength of the electric field at the position of the charge can be determined by measured a force acting on a small test charge at rest.

Next, if a small test charge moves with the velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, it is known to experience the magnetic force

$$\mathbf{F}_m = q(\mathbf{v} \times \mathbf{B}). \tag{2.6}$$

Example 2.1. Determine the components of the charge velocity $\mathbf{v}_{\|}$ and $\mathbf{v}_{\perp}$, parallel and perpendicular to the magnetic field B, respectively.
Solution. Introduce a unit vector along $\mathbf{B}$, $\mathbf{b} = \mathbf{B}/B$. *It follows from the definition of the dot product that the projection of* $\mathbf{v}$ *onto* $\mathbf{b}$ *is* $\mathbf{v} \cdot \mathbf{b}$. *Hence the vector projection along* $\mathbf{b}$ *is* $\mathbf{v}_{\|} = (\mathbf{b} \cdot \mathbf{v})\mathbf{b}$. *Consequently,* $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\|} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{b})\mathbf{b}$. *Note that* $\mathbf{v}_{\perp} \cdot \mathbf{b} \equiv 0$.

On taking a cross product of both sides of Eq. (2.6) with $\mathbf{v}_{\perp}$ we obtain

$$\mathbf{F}_m \times \mathbf{v}_{\perp} = -qB\mathbf{v}_{\perp} \times (\mathbf{v} \times \mathbf{b}) = -q\mathbf{v}(\mathbf{v}_{\perp} \cdot \mathbf{b}) + q\mathbf{B}(\mathbf{v} \cdot \mathbf{v}_{\perp}) = q\mathbf{B}(\mathbf{v} \cdot \mathbf{v}_{\perp}).$$

Note also that $\mathbf{v}_{\perp} \cdot \mathbf{v} = v^2 - v_{\|}^2$. Thus we arrive at the expression for the magnetic field

$$\boxed{\mathbf{B} = \frac{(\mathbf{F}_m \times \mathbf{v}_{\perp})}{q(v^2 - v_{\|}^2)}}. \tag{2.7}$$

Thus, we can determine $\mathbf{B}$ by measuring the force on a moving charge and the charge velocity. Note that Eq. (2.7) is indeterminate whenever $v = v_{\|}$, because in this case the force equals to zero according to Eq. (2.6). So the charge velocity should have a component at an angle to the magnetic field to unambiguously determine the latter. We note that Eqs. (2.5) and (2.7) serve as the operational definitions of $\mathbf{E}$ and $\mathbf{B}$, respectively. The latter characterize the strength of electric and magnetic interactions at a given point in space–described by a position radius vector **r**– and hence are **local measures** of the electromagnetic interactions in a given system. Thus, $\mathbf{E}$ and $\mathbf{B}$ are functions of the space coordinate; in general, they can also, vary with time,

$$\mathbf{E} = \mathbf{E}(\mathbf{r}, t) \qquad \text{and} \qquad \mathbf{B} = \mathbf{B}(\mathbf{r}, t).$$

If a test charge is moving in both electric and magnetic fields, which are, in general, functions of time, it experiences the combined Lorentz force,

$$\boxed{\mathbf{F}_L = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}},$$

and the electric and magnetic fields can be thought of as components of a common entity called the **electromagnetic field**.

Example 2.2. A point charge Q with a velocity $\mathbf{v} = v_0\mathbf{a}_x$ enters a region of space with a uniform magnetic field. The magnetic flux density in the region is $\mathrm{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. What E should exist in the region for the charge to proceed without change of its velocity.

Solution. Assuming $\mathbf{E} = E_x\mathbf{a}_x + E_y\mathbf{a}_y + E_z\mathbf{a}_z$ and writing down the second law of Newton in components, we arrive at the equations,

$$mv_x' = QE_x + Q(v_yB_z - B_yv_z),$$

$$mv_y' = QE_y + Q(v_zB_x - B_zv_x),$$

and

$$mv_z' = QE_z + Q(v_xB_y - B_xv_y).$$

Here the prime stands for a time derivative. The charge will proceed with the same velocity if all components of the acceleration vanish at all times, i. e, $v_x' = v_y' = v_z' = 0$. Since at $t = 0$ $v_y = v_z = 0$, it follows that $E_x = 0$, $E_y = B_zv_0$ and $E_z = -v_0B_y$. Thus, $\underline{\mathbf{E} = v_0(B_z\mathbf{a}_y - B_y\mathbf{a}_z)}$.

2.2 Electromagnetic fields in materials

The response of a material to an applied electric field depends on whether the material has free electrons and therefore can conduct currents or not. Materials of the first kind are called **conductors** whereas the rest are known as **dielectrics**.

In conductors, the electrons are free to move and their motion past heavy ions of a crystal lattice constitutes a **conduction current**. One can introduce a local quantity characterizing the current, the current density **J** measured in ampères per square meter, which is just a current per unit cross section of a conductor. The total current is then

$$\boxed{I = \int d\mathbf{S}\cdot\mathbf{J}}, \tag{2.8}$$

where $d\mathbf{S} = \mathbf{a}_n dS$ is an oriented elementary surface, $\mathbf{a}_n$ being a **unit normal** to the surface as is indicated in Fig. 2.2.

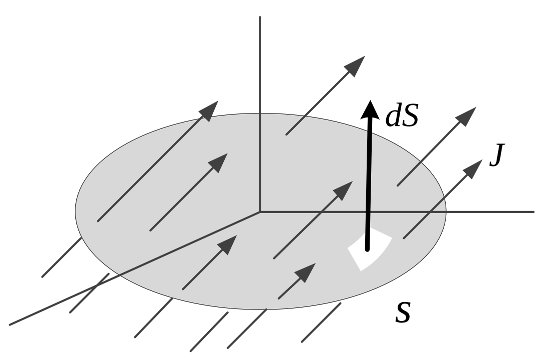

Figure 2.2: Illustrating the current density definition.

The integral on the r.h.s of Eq. (2.8) is an example of a **flux** of the vector field–in this instance $\mathbf{J}$–through an **open** surface S.

Example 2. 3. Show that $\mathbf{J} = \rho_v \mathbf{v}$ where ρ_v is the volume charge density and $\mathbf{v}$ is the drift velocity of charge carriers.

Solution. Consider a small volume element $dv = d\mathbf{S} \cdot \mathbf{v}dt$. The amount of charge inside a cylindrical volume of height $\mathbf{v} \cdot \mathbf{a}_n dt$ with a finite cross-section S is $dQ = \int_S (d\mathbf{S} \cdot \mathbf{v}) dt \rho_v$. By definition, the current through the cross-section S is then $I = dQ/dt = \int_S d\mathbf{S} \cdot \mathbf{v}\rho_v = \int_S d\mathbf{S} \cdot \mathbf{J}$. It follows that $\mathbf{J} = \rho_v \mathbf{v}$.

The current density is related to the electric field via the **local form** of Ohm's law,

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}, \tag{2.9}$$

where σ is called the electric conductivity, measured in siemens per meter, S/m.

In the dielectrics, the electrons are bound to nuclei forming neutral atoms. The application of an external electric field, however, causes spatial displacement of negatively charged electron clouds away from positively charged nuclei; the latter being so heavy that they remain immobile. The medium is then said to be polarized. This process is illustrated in Fig. 2.3.

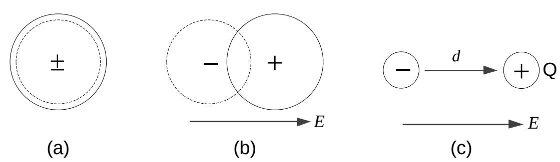

Figure 2.3: Illustrating the polarization of a nonpolar dielectric.

The polarization can be quantitatively described in terms of individual atom dipole moments.

Definition. An individual dipole moment vector $\mathbf{p}$ is defined as the product of an electron cloud charge and a position vector from the nucleus to the center of the electron cloud. For instance, for an atom having just one bound electron, $\mathbf{p} = -e\mathbf{r}$.

The dielectrics with the atoms that have no dipole moments in the absence of the applied field are called **nonpolar**. Alternatively, the medium atoms of **polar** dielectrics can have nonzero dipole moments even in the absence of $\mathbf{E}$, but they are randomly oriented. As the external electric field is applied, though, the dipoles align along the field resulting in the medium polarization.

Regardless of a specific polarization origin, we can define a **macroscopic** polarization field.

Definition. The **polarization field** $\mathbf{P}(\mathbf{r}, t)$ is a dipole moment per unit volume at the position $\mathbf{r}$ within a polarized medium.

The polarized medium alters (reduces) the external electric field $\mathbf{E}$ such that the effective field inside the medium is described in terms of the **electric flux density** $\mathbf{D}$,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}}, \tag{2.10}$$

where $\epsilon_0 = 8.854 \times 10^{-12}$ farad per meter (F/m) is the so-called dielectric permittivity of free space. Eq. (2.10) works for any dielectric; throughout this course we will be dealing with **linear, homogeneous, isotropic** dielectrics for which $\mathbf{P}$ is linearly related to $\mathbf{E}$ viz.,

$$\boxed{\mathbf{P} = \epsilon_0 \chi_{\rm e} \mathbf{E}}, \tag{2.11}$$

where $\chi_{\rm e}$ is the **electric susceptibility**. It follows from Eqs. (2.10) and (2.11) that

$$\boxed{\mathbf{D} = \epsilon_0(1 + \chi_{\rm e})\mathbf{E} = \epsilon_0 \epsilon_{\rm r} \mathbf{E} = \epsilon \mathbf{E}}. \tag{2.12}$$

Here ϵ is the **dielectric permittivity** of the medium, and $\epsilon_{\rm r}$ is a **dielectric constant** (relative permittivity). Note that while ϵ has the same units as ϵ_0, i.e, farads per meter, $\epsilon_{\rm r}$ is dimensionless.

In reality, of course, there are no ideal conductors, nor are there ideal dielectrics. Real materials have both bound and free (conducting) electrons and are characterized by finite ϵ and σ. The distinction in the behavior of dielectrics and conductors depends on the frequency of the applied time-harmonic electric field, $\mathbf{E}(t) = \mathbf{E}_\omega e^{-j\omega t}$. To drive this point home, we develop a simple classical model of matter response to an external time-harmonic field. In this model–which works well for linear homogeneous isotropic materials–atoms are treated as simple harmonic oscillators. That is, bound electrons are assumed to be attached to the nuclei by "springs" which provide "restoring" forces proportional to the electron displacement from the nucleus. Physically, the restoring force experienced by an electron is due to its attraction to the nucleus. Assume that each atom has Z bound electrons. Assume further that there are $f_{\rm s}$ electrons per atom having the binding frequency $\omega_{\rm s}$ which corresponds to a particular type of the "spring". The quantities $\{f_{\rm s}\}$ are referred to as the oscillator strengths.

Whenever an electron having the binding frequency ω_s is displaced by the displacement vector $\mathbf{r}_{\rm s}$ in response to the external electric field, it experiences three forces: the restoring force, $\mathbf{F}_{\rm r} = -m\omega_{\rm s}^2 \mathbf{r}_{\rm s}$, the damping force, $\mathbf{F}_{\rm d} = -2m\gamma_s \dot{\mathbf{r}}_{\rm s}$–where $\gamma_{\rm s}$ is a phenomenological damping constant–and the force due to the external electric field, $\mathbf{F}_{\rm e} = -e\mathbf{E}_\omega e^{-j\omega t}$.

The equation of electron motion (second law of Newton) is then

$$m\ddot{\mathbf{r}}_{\rm s} = -m\omega_{\rm s}^2 \mathbf{r}_{\rm s} - 2m\gamma_{\rm s}\dot{\mathbf{r}}_{\rm s} - e\mathbf{E}_\omega e^{-j\omega t}. \tag{2.13}$$

Here each "dot" stands for a time derivative. We seek a driven solution to Eq. (2.13) in the form,

$$\mathbf{r}_{\rm s}(t) = \mathbf{r}_{{\rm s}\omega} e^{-j\omega t}. \tag{2.14}$$

It follows from Eqs. (2.13) and (2.14) that the electron displacement amplitude is

$$\mathbf{r}_{{\rm s}\omega} = -\frac{e\mathbf{E}_\omega}{m(\omega_{\rm s}^2 - \omega^2 - 2j\omega\gamma_{\rm s})}, \tag{2.15}$$

implying that

$$\mathbf{r}_{\rm s}(t) = -\frac{e\mathbf{E}(t)}{m(\omega_{\rm s}^2 - \omega^2 - 2j\omega\gamma_{\rm s})}. \tag{2.16}$$

The induced individual dipole moment of the electron of this type will be $\mathbf{p}_{\rm s} = -e\mathbf{r}_{\rm s}$. Next, if there are N atoms per unit volume, the induced polarization is

$$\mathbf{P} = N\sum_{\rm s} f_s\mathbf{p}_{\rm s} = -Ne\sum_{\rm s} f_{\rm s}\mathbf{r}_{\rm s} = \frac{Ne^2}{m}\sum_{\rm s}\frac{f_{\rm s}\mathbf{E}}{(\omega_{\rm s}^2 - \omega^2 - 2j\omega\gamma_{\rm s})}. \tag{2.17}$$

Note that the oscillator strengths satisfy the so-called sum rule

$$\sum_{\rm s} f_{\rm s} = Z. \tag{2.18}$$

On comparing Eqs. (2.11), (2.12) and (2.17), we infer that

$$\epsilon(\omega) = \epsilon_0\left[1 + \frac{Ne^2}{m}\sum_{\rm s}\frac{f_{\rm s}}{(\omega_{\rm s}^2 - \omega^2 - 2j\omega\gamma_{\rm s})}\right], \tag{2.19}$$

which provides a classical expression for the dielectric permittivity of materials as a function of frequency of the applied electric field. Here the imaginary part of ϵ describes absorption of electromagnetic waves as we will see in Chapter 3.

Let us now explore what happens if the frequency of the applied electric field is close to a particular resonant frequency of the material. For the sake of clarity, let that be the lowest bound frequency of the dielectric, $\omega_0 \neq 0$, i.e, $\omega \approx \omega_0$. In this case, we can single out the resonant term in Eq. (2.19) implying that

$$\epsilon(\omega) = \epsilon_{\rm NR}(\omega) + \frac{\epsilon_0 Ne^2 f_0}{m}\frac{1}{(\omega_0^2 - \omega^2 - 2j\omega\gamma_0)}. \tag{2.20}$$

As typically $\gamma_s \ll \omega_s$, the contribution to the permittivity due to non-resonant terms, $\epsilon_{\rm NR}$ is a purely real and only weakly frequency dependent. It can be expressed as

$$\epsilon_{\rm NR}(\omega) \simeq \epsilon_0\sum_{s\neq 0}\frac{Ne^2 f_{\rm s}/m}{(\omega_{\rm s}^2 - \omega^2)}. \tag{2.21}$$

Notice that close to resonance, we can approximate

$$-\omega^2 + \omega_0^2 - 2j\gamma_0\omega \simeq 2\omega(\omega_0 - \omega - j\gamma_0) \simeq 2\omega_0(\omega_0 - \omega - j\gamma_0). \tag{2.22}$$

It can be inferred from Eqs. (2.21) and (2.22) that the electric susceptibility near optical resonance can be represented as

$$\chi_{\rm e}(\omega) = \chi_{\rm e}'(\omega) + j\chi_{\rm e}''(\omega), \tag{2.23}$$

where

$$\chi_{\rm e}'(\omega) = \chi_{\rm NR}(\omega) + \frac{Ne^2 f_0}{2\epsilon_0 m\omega_0}\left[\frac{\omega - \omega_0}{(\omega - \omega_0)^2 + \gamma_0^2}\right], \tag{2.24}$$

and

$$\chi_{\rm e}''(\omega) = \frac{Ne^2 f_0}{2\epsilon_0 m\omega_0}\left[\frac{\gamma_0}{(\omega - \omega_0)^2 + \gamma_0^2}\right]. \tag{2.25}$$

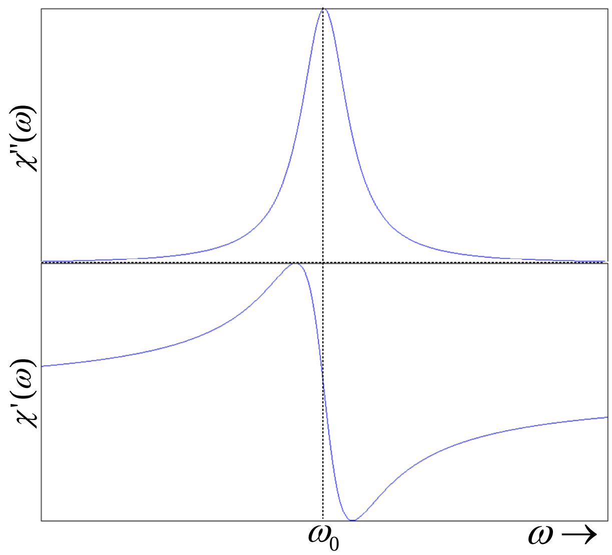

Figure 2.4: Imaginary (top) and real (bottom) parts of linear susceptibility as functions of frequency near resonance.

The real and imaginary parts of χ_{e} are sketched as functions of the frequency in Fig. 2.5.

The difference between conductors and dielectrics can then be attributed to the presence of free electrons in the former. Indeed, by looking into the low-frequency limit, we notice that for pure dielectrics the lowest bound frequency must be nonzero, while conductors can have a fraction of electrons, f_0, say, that have $\omega_0 = 0$; those are essentially free electrons. Consequently, the dielectric permittivity of conductors is given by the expression

$$\epsilon_{\mathrm{c}}(\omega) = \epsilon_{\mathrm{b}}(\omega) + j\frac{\epsilon_0 N f_0 e^2}{m\omega(2\gamma_0 - j\omega)}, \tag{2.26}$$

where ϵ_b is the overall contribution of the bound electrons with $\omega_s \neq 0$. Since free electrons can conduct currents, we can use Eq. (2.16) and Example 2.3, to work out the current density,

$$\mathbf{J} = -Nef_0\dot{\mathbf{r}}_0 = \frac{Nf_0e^2}{m(2\gamma_0 - j\omega)}\mathbf{E}. \tag{2.27}$$

On comparing Eqs. (2.9) and (2.27), we infer the expression for the conductivity,

$$\sigma(\omega) = \frac{Nf_0e^2}{m(2\gamma_0 - j\omega)}. \tag{2.28}$$

It is seen from Eq. (2.28) that in the dc limit $\omega \to 0$,

$$\sigma \to \frac{Nf_0e^2}{2m\gamma_0} = \sigma_0, \tag{2.29}$$

the conductivity is real, describing dc currents. In view of Eq. (2.29), the expression for σ can be cast into the form

$$\sigma(\omega) = \frac{\sigma_0}{1 - j\omega\tau}, \tag{2.30}$$

where $\tau = 1/2\gamma_0$ is a characteristic time for current relaxation in conductors.

Next, comparing Eqs. (2.26) and (2.28), we can express the former as

$$\epsilon_{\rm c}(\omega) = \epsilon_{\rm b}(\omega) + j(\sigma/\omega). \tag{2.31}$$

Eq. (2.31) implies that losses in real conductors/metals come in two guises: the absorption of electromagnetic waves by bound electrons–which is described by the imaginary part of ϵ_b–and ohmic losses due to generating electric currents as described by the second term on the r.h.s.

Finally, we note that at high frequencies, $\omega \gg \max(\omega_s)$, dielectrics and conductors respond to the applied electric field the same wave. In this limit, we can neglect all $\{\omega_s\}$ and $\{\gamma_s\}$ in the denominator of Eq. (2.19), leading to

$$\epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right), \tag{2.32}$$

where we used Eq. (2.18) and introduced the plasma frequency

$$\omega_{\rm p} = \sqrt{\frac{NZe^2}{\epsilon_0 m}}. \tag{2.33}$$

The phenomenological treatment of macroscopic medium response to the magnetic field parallels that we just presented. An external magnetic field causes the medium **magnetization**: the atomic magnetic moments align along the applied field causing a finite macroscopic average dipole moment density. The latter called **magnetization** $\mathbf{M}$, and is a magnetic analog of $\mathbf{P}$. By analogy, the **magnetic field intensity** $\mathbf{H}$ within the magnetized medium can be determined as

$$\boxed{\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}}. \tag{2.34}$$

Eq. (2.34) holds true for any medium. Here,

$$\boxed{\mu_0 = 4\pi \times 10^{-7}}, \ \text{H/m}, \tag{2.35}$$

is known as the **free space permeability**. In the case of a **linear, homogeneous, isotropic** magnetic, we obtain

$$\boxed{\mathbf{M} = \chi_{\rm m}\mathbf{H}}, \tag{2.36}$$

where $\chi_{\rm m}$ is the **magnetic susceptibility**, implying that

$$\boxed{\mathbf{B} = \mu_0(1 + \chi_{\rm m})\mathbf{H} = \mu_0\mu_{\rm r}\mathbf{H} = \mu\mathbf{H}}. \tag{2.37}$$

Here μ and $\mu_{\rm r}$ are the **magnetic permeability** and **relative magnetic permeability** of the medium. Note that while $\mathbf{E}$ and $\mathbf{B}$ are directly related to measurable quantities, the forces on charges, $\mathbf{D}$ and $\mathbf{H}$ are auxiliary fields.

2.3 Global or integral form of Maxwell's equations

The first two Maxwell's equations are mathematical expressions of the fact that static charges are sources of the electric field and there are no static magnetic charges. In particular, the first Maxwell equation–also known as the electric Gauss's law–states that the total flux of $\mathbf{D}$ through any **closed** surface S is equal to the total **enclosed charge**,

$$\boxed{\oint_S d\mathbf{S} \cdot \mathbf{D} = Q_{enc} = \int_v dv \rho_v}. \tag{2.38}$$

Here the circle around the integral implies that the surface for the surface integration must be closed. The choice of the oriented elementary surface $d\mathbf{S} = \mathbf{a}_n dS$ used on the l.h.s of Eq. (2.38) is ambiguous as the unit normal can be directed either inside or outside the volume enclosed by S. By convention, we choose $\mathbf{a}_n$ to be the **outward** unit normal as is indicated in Fig. 2.5. Also ρ_v is the volume density–in C/m^3–of the charge inside S.

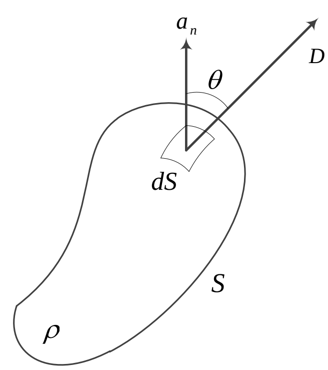

Figure 2.5: Outward unit normal to a closed surface S.

Since there are no static magnetic charges, the second Maxwell equation (magnetic Gauss's law) states that

$$\boxed{\oint_S d\mathbf{S} \cdot \mathbf{B} = 0}. \tag{2.39}$$

It is now clear from Eqs. (2.38) and (2.39) why $\mathbf{D}$ and $\mathbf{B}$ are referred to as the electric and magnetic flux densities, respectively.

In the Cartesian coordinate system, the infinitesimally small surface and volume elements required in Eqs. (2.38) and (2.39) can be expressed as

$$d\mathbf{S} = \begin{cases} dydz\mathbf{a}_x \\ dxdz\mathbf{a}_y \\ dxdy\mathbf{a}_z \end{cases}$$

and

$$dv = dxdydz.$$

The surface element calculation is illustrated in Fig. 2.6.

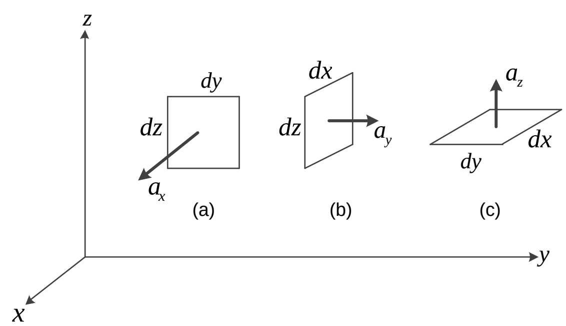

Figure 2.6: Illustrating the elementary surfaces in the Cartesian coordinates.

The third Maxwell equation, or the Faraday's law, relates the electric field **circulation** around any **closed path** C with the time rate of change of the magnetic flux through an open surface S bounded by the path,

$$\boxed{\underbrace{\oint_C d\mathbf{l}\cdot\mathbf{E}}_{\text{emf}} = -\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{B}}. \tag{2.40}$$

In the circulation integral on the l.h.s. of Eq. (2.40), $d\mathbf{l}$ is an **oriented** infinitesimally small path element which can be expressed in the Cartesian coordinates as

$$d\mathbf{l} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z.$$

The fourth Maxwell equation–sometimes referred to as Ampère's law–links the circulation of the magnetic field along a closed path with the flux of the overall current–conduction current plus displacement current–through an open surface S rimmed by the path,

$$\boxed{\underbrace{\oint_C d\mathbf{l}\cdot\mathbf{H}}_{\text{mmf}} = \underbrace{\int_S d\mathbf{S}\cdot\mathbf{J}}_{\text{conduction}} + \underbrace{\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{D}}_{\text{displacement}}}. \tag{2.41}$$

To summarize, the first two Maxwell equations state the existence of static electric and nonexistence of static magnetic charges. The third one implies that a time-varying magnetic field can induce an electric field with the electromotive force (emf),

$$\boxed{\mathcal{E}_{\text{emf}} \equiv \oint_C d\mathbf{l}\cdot\mathbf{E}}, \tag{2.42}$$

in a given closed loop determined by the time rate of change of the magnetic flux through the loop, that is

$$\boxed{\mathcal{E}_{\text{emf}} = -\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{B}}. \tag{2.43}$$

The fourth equation tells us that conduction and/or displacement currents generate a magnetic field with the magnetomotive force (mmf),

$$\mathcal{E}_{\text{mmf}} \equiv \oint_C d\mathbf{l}\cdot\mathbf{H}, \tag{2.44}$$

in a closed loop determined by the time rate of change of the total current flux through the loop. In particular, in the absence of conduction currents $\mathbf{J} = 0$, the time-varying electric fields can generate magnetic fields. Thus the propagation of electromagnetic waves in source-free space, ($\rho_v = 0$, $\mathbf{J} = 0$) is a direct consequence of Eqs. (2.40) and (2.41). Note also that the displacement current is a fictitious current that has to do with time-varying electric fields.

Example 2. 4. A magnetic flux density is given by $\mathbf{B} = \mathbf{a}_y B_0/x$ Wb/m^2, where B_0 is a constant. A rigid rectangular loop is situated in the xz-plane with the corners at the points (x_0, z_0), (x_0, z_0+b), (x_0+a, z_0+b), (x_0+a, z_0). If the loop is moving with the velocity $\mathbf{v} = v_0\mathbf{a}_x$, determine the induced emf.

Solution. At the time t the corners of the loop will be at the points (x_0+vt, z_0), (x_0+vt, z_0+b), (x_0+a+vt, z_0+b), (x_0+a+vt, z_0). Using the Faraday's law, $\mathcal{E}_{emf} = \oint_C d\mathbf{l}\cdot\mathbf{E} = -\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{B}$. In our case, $d\mathbf{S} = dxdz\mathbf{a}_y$ implying that

$$\int_S d\mathbf{S}\cdot\mathbf{B} = B_0\int_{z_0}^{z_0+b} dz\int_{x_0+vt}^{x_0+a+vt}\frac{dx}{x} = B_0 b\ln\frac{x_0+a+vt}{x_0+vt}.$$

It then follows that

$$\underline{\mathcal{E}_{efm} = B_0 bv\left(\frac{1}{x_0+vt} - \frac{1}{x_0+a+vt}\right)}.$$

Example 2. 5. Solve the previous problem for a stationary loop in the time-varying magnetic field $\mathbf{B} = \mathbf{a}_y(B_0/x)\cos\omega t$ Wb/m^2.

Solution. If the loop is at rest, by analogy with the previous example,

$$\begin{aligned}\mathcal{E}_{emf} &= -\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{B} = \omega B_0\sin\omega t\int_{z_0}^{z_0+b} dz\int_{x_0}^{x_0+a}\frac{dx}{x}\\ &= \omega b B_0\sin\omega t\ln\frac{x_0+a}{x_0}.\end{aligned} \tag{2.45}$$

Thus,

$$\underline{\mathcal{E}_{emf} = \omega b B_0\sin\omega t\ln\frac{x_0+a}{x_0}}.$$

Example 2. 6. Assume that the loop in Example 2.3 moves with the velocity $\mathbf{v} = v_0\mathbf{a}_x$ in the time-varying field $\mathbf{B} = \mathbf{a}_y(B_0/x)\cos\omega t$ Wb/m^2, find the induced emf.

Solution. In this case, the loop moves and the magnetic flux density changes with time such that

$$\mathcal{E}_{emf} = -B_0 \frac{d}{dt} \cos\omega t \int_{z_0}^{z_0+b} dz \int_{x_0+vt}^{x_0+a+vt} \frac{dx}{x} = -B_0 b \frac{d}{dt} \left(\cos\omega t \ln \frac{x_0+a+vt}{x_0+vt} \right). \tag{2.46}$$

Doing the derivative, we obtain

$$\mathcal{E}_{emf} = B_0 b \left[\omega \sin\omega t \ln \frac{x_0+a+vt}{x_0+vt} + v\cos\omega t \times \left(\frac{1}{x_0+vt} - \frac{1}{x_0+a+vt} \right) \right]. \tag{2.47}$$

2.4 Boundary conditions in electro-magnetics

We consider an interface separating two media. The boundary conditions linking the electromagnetic fields on both sides of the interface can be derived from the Maxwell equations in the integral form. To this end, introduce a set of three mutually orthogonal unit vectors: the outward unit normal pointing into medium 2, $\mathbf{a}_{n12}$, the unit tangential vector $\mathbf{a}_\tau$ and unit bi-normal vector $\mathbf{a}_b$ such that (see Fig. 2. 7.),

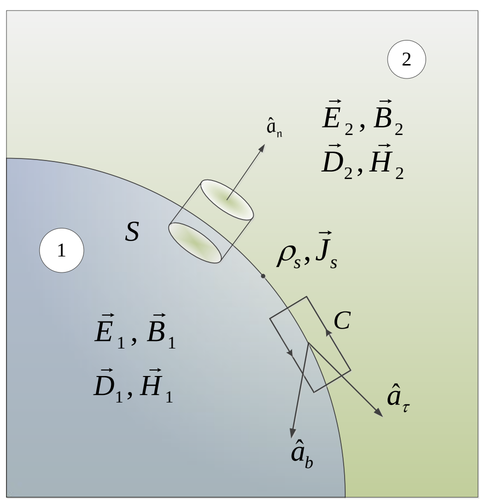

Figure 2.7: Electromagnetic fields at the interface between the two homogeneous media.

$$\mathbf{a}_b = \mathbf{a}_{n12} \times \mathbf{a}_\tau. \tag{2.48}$$

Let us decompose all fields into normal and tangential components to the interface such that

$$\mathbf{E} = \mathbf{E}_n + \mathbf{E}_\tau, \qquad \mathbf{B} = \mathbf{B}_n + \mathbf{B}_\tau, \tag{2.49}$$

with the similar expressions for $\mathbf{D}$ and $\mathbf{H}$. It can be inferred from geometry that

$$\mathbf{E}_n = \mathbf{a}_n(\mathbf{E}\cdot\mathbf{a}_n) \tag{2.50}$$

and

$$\mathbf{E}_\tau = \mathbf{E} - \mathbf{a}_n(\mathbf{E}\cdot\mathbf{a}_n) = \mathbf{a}_n \times (\mathbf{E}\times\mathbf{a}_n), \tag{2.51}$$

where "bac-cab" rule was used on the r.h.s of Eq. (2.51).

Applying the electric Gauss's law, Eqs. (2.38), to the cylindrical Gaussian pillbox S shown in Fig. 2.7 and taking the limit of a very shallow pillbox, we obtain

$$\oint_S dS\,\mathbf{a}_{n12}\cdot\mathbf{D} = \mathbf{a}_{n12}\cdot(\mathbf{D}_2 - \mathbf{D}_1)\Delta S = \int dv\rho_v = \rho_S\Delta S, \tag{2.52}$$

where ρ_s is the surface charge density on the interface. It follows at once from Eq. (2.52) that

$$\boxed{\mathbf{a}_{n12}\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \rho_S}. \tag{2.53}$$

By the same token,

$$\boxed{\mathbf{a}_{n12}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = 0}, \tag{2.54}$$

because there are no magnetic charges. Eqs. (2.53) and (2.54) relate the normal components of the electric and magnetic flux densities on both sides of the interface.

Applying now the Faraday law (2.40) to the Stockesian loop C, we obtain in the limit of a very small loop the expression

$$\oint_C d\mathbf{l}\cdot\mathbf{E} = \mathbf{a}_\tau\cdot(\mathbf{E}_2 - \mathbf{E}_1)\Delta l = 0, \tag{2.55}$$

since $\partial_t\mathbf{B}$ is finite on the surface of C and the surface area vanishes as we shrink the loop sides. Thus,

$$\mathbf{E}_{2\tau} - \mathbf{E}_{1\tau} = 0, \tag{2.56}$$

or, alternatively, with the help of Eq. (2.51),

$$\mathbf{a}_{n12}\times[\mathbf{a}_{n12}\times(\mathbf{E}_2 - \mathbf{E}_1)] = 0, \tag{2.57}$$

implying for an arbitrary point of the surface that

$$\boxed{\mathbf{a}_{n12}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0}. \tag{2.58}$$

At the same time, the Ampére equation tells us that

$$\begin{aligned}
&\oint_C d\mathbf{l}\cdot\mathbf{E} = \mathbf{a}_\tau\cdot(\mathbf{H}_2 - \mathbf{H}_1)\Delta l = (\mathbf{a}_b\times\mathbf{a}_{n12})\cdot(\mathbf{H}_2 - \mathbf{H}_1)\Delta l = \\
&= \mathbf{a}_b\cdot[\mathbf{a}_{n12}\times(\mathbf{H}_2 - \mathbf{H}_1)]\Delta l = \int_{S_C} dS\mathbf{a}_b\cdot\left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right) \\
&= \mathbf{a}_b\cdot\mathbf{J}_s\Delta l
\end{aligned} \tag{2.59}$$

where $\mathbf{J}_s$ is the surface current at the interface. It can be inferred from (2.59) that

$$\boxed{\mathbf{a}_{n12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s}. \tag{2.60}$$

Eqs. (2.53), (2.54), (2.58) and (2.60) constitute the general boundary conditions in the electromagnetic theory.

2.5 Local or differential form of Maxwell's equations

The Maxwell equations can also be cast into a differential (local) form in which they pertain to any spatial point within a given region of space. Although local Maxwell's equations are less physically intuitive, they are more suitable to mathematically describe versatile electromagnetic problems. We begin by introducing local measures of the vector field flux and circulation, the flux and circulation densities, or the **divergence** and **curl** of the vector field.

Definition. The divergence of a vector field $\mathbf{A}$ at a given point is the net outward flux of $\mathbf{A}$ per unit volume at the point. Mathematically,

$$\mathrm{div}\mathbf{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S d\mathbf{S} \cdot \mathbf{A}}{\Delta v}. \tag{2.61}$$

It is known from the vector calculus that the divergence can also be written in terms of the Del operator, denoted ∇, as

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A}. \tag{2.62}$$

In the Cartesian coordinates, the latter is defined as

$$\nabla = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}.$$

And since $\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z$, we conclude that

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{2.63}$$

In practice, the following divergence theorem is often handy in working out fluxes of vector fields through closed surfaces.

Divergence Theorem. The flux of a vector field through a closed surface equals the integral of the vector field divergence over the volume enclosed by the surface,

$$\oint_S d\mathbf{S} \cdot \mathbf{A} = \int_v dv\, \nabla \cdot \mathbf{A}. \tag{2.64}$$

We are now in a position to express the first two Maxwell's equations in the local form. Applying the divergence theorem to the l.h.s of Eq. (2.38), we obtain

$$\oint_S d\mathbf{S} \cdot \mathbf{D} = \int_v dv\, \nabla \cdot \mathbf{D} = \int_v dv \rho_v. \tag{2.65}$$

It can be inferred from Eq. (2.65) that

$$\int_v dv(\nabla \cdot \mathbf{D} - \rho_v) = 0. \tag{2.66}$$

Since the integral equation (2.66) holds for any volume, we conclude that the integrand must be equal to zero at any point within the volume,

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v}. \tag{2.67}$$

By the same token,

$$\boxed{\nabla \cdot \mathbf{B} = 0}. \tag{2.68}$$

Next, we introduce the curl of a vector field as

Definition. The curl of a vector field $\mathbf{A}$ at a given point is a vector with a magnitude equal to the maximum net circulation of $\mathbf{A}$ per unit area at the point. The curl is directed along a unit normal to the infinitesimal area around the point which is oriented to maximize the curl. The unit normal is chosen to conform to the right-hand rule: whenever the fingers of your right hand follow the direction of $d\mathbf{l}$ along the area border, your thumb points in the direction of the unit normal.

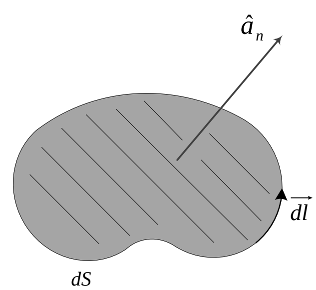

Figure 2.8: Illustrating the choice of unit normal in curl evaluation.

Mathematically, curl can be defined as

$$\text{curl}\mathbf{A} = \nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \frac{\oint_C d\mathbf{l} \cdot \mathbf{A}}{\Delta S}. \tag{2.69}$$

In the Cartesian coordinates, the curl can be expressed as

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{2.70}$$

The following curl theorem is often useful in determining the circulation of a vector field around a closed loop.

Curl Theorem. The circulation of a vector field along a closed path is equal to the flux of the vector field curl through an open surface bounded by the path,

$$\oint_C d\mathbf{l}\cdot\mathbf{A} = \int_S d\mathbf{S}\cdot\nabla(\times\mathbf{A}). \tag{2.71}$$

With the aid of Eq. (2.71) and assuming that the loop C is stationary, one can transform the third Maxwell equation as

$$\oint_C d\mathbf{l}\cdot\mathbf{E} = \int_S d\mathbf{S}\cdot(\nabla\times\mathbf{E}) = -\int_S d\mathbf{S}\cdot\frac{\partial\mathbf{B}}{\partial t},$$

implying that locally

$$\boxed{\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}}. \tag{2.72}$$

Similarly,

$$\oint_C d\mathbf{l}\cdot\mathbf{H} = \int_S d\mathbf{S}\cdot(\nabla\times\mathbf{H}) = \int_S d\mathbf{S}\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right).$$

It can then be inferred at once that

$$\boxed{\nabla\times\mathbf{H} = \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}}. \tag{2.73}$$

Here the second term on the rhs is the **displacement current** (density) defined as

$$\boxed{\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t}}. \tag{2.74}$$

Example 2.7. Given $\mathbf{B} = (10^{-7}/3)\cos(6\pi\times 10^8 t - 2\pi z)\mathbf{a}_y$ Wb/m^2 in free space, find E.

Solution. In free space, $\mathbf{B} = \mu_0\mathbf{H}$ and $\mathbf{D} = \epsilon_0\mathbf{E}$. Hence, $\mathbf{H} = (1/12\pi)\cos(6\pi\times 10^8 t - 2\pi z)\mathbf{a}_y$ A/m. Use Ampère's law (2.73) with $\mathbf{J} = 0$,

$$\nabla\times\mathbf{H} = \epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{2.75}$$

we can work out the l.h.s,

$$\nabla\times\mathbf{H} = \mathbf{a}_x\frac{1}{6}\sin(6\pi\times 10^8 t - 2\pi z) = \mathbf{a}_x\epsilon_0\frac{\partial E_x}{\partial t} \tag{2.76}$$

It follows that the electric field must have only x-component. We can infer from Eqs. (2.75) and (2.76) that $\mathbf{E}$ should have the form

$$\mathbf{E} = \mathbf{a}_x A\cos(6\pi\times 10^8 t - 2\pi z), \tag{2.77}$$

where A is an unknown constant. On substituting from Eq. (2.77) into Eq. (2.75) and using Eq. (2.76), we obtain

$$\frac{1}{6}\sin(6\pi\times 10^8 t - 2\pi z) = 6\pi\epsilon_0\times A\times 10^8\sin(6\pi\times 10^8 t - 2\pi z),$$

implying that $A = 10^{-8}/36\pi\epsilon_0$. As $\epsilon_0 = 10^{-9}/36\pi$, we arrive at $A = 10$. Thus, $\underline{\mathbf{E} = 10\cos(6\pi\times 10^8 t - 2\pi z)\mathbf{a}_x}$ V/m.

2.6 Conservation laws in electromagnetic theory

We will now examine two important conservation laws encountered in the electromagnetic theory: the **charge** and electromagnetic **energy** conservation. While the former is a fundamental law of nature, **independent** of Maxwell's equations, the latter is their direct consequence.

The charge conservation law states that charges cannot be created, nor can they be annihilated. In a global sense, this statement implies that an overall charge within any finite volume must be conserved. Therefore the time rate of change of the charge within the volume is equal to the current flux through the surface enclosing the volume,

$$\boxed{\frac{d}{dt}\int dv\rho_v = -\oint d\mathbf{S}\cdot\mathbf{J}}. \tag{2.78}$$

The minus sign in Eq. (2.78) indicates the fact that the charge within the volume decreases (increases) if the current flows outside (inside) the volume, see Fig. 2.9.

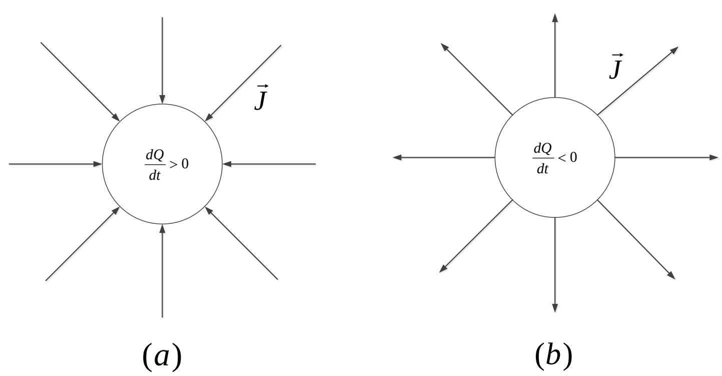

Figure 2.9: Illustrating the charge conservation law.

Assuming the volume in Eq. (2.78) is at rest, the local form of the charge conservation law follows from Eq. (2.78) on the application of the divergence theorem to the r.h.s,

$$\frac{d}{dt}\int dv\rho_v = \int_v dv\frac{\partial\rho_v}{\partial t} = -\oint d\mathbf{S}\cdot\mathbf{J} = -\int dv\,\nabla\cdot\mathbf{J}. \tag{2.79}$$

It follows at once from Eq. (2.79) that

$$\int_v\left(\frac{\partial\rho_v}{\partial t} + \nabla\cdot\mathbf{J}\right) = 0, \tag{2.80}$$

implying the local **continuity equation**

$$\boxed{\frac{\partial\rho_v}{\partial t} + \nabla\cdot\mathbf{J} = 0}. \tag{2.81}$$

Example 2.8. Given $\mathbf{J} = \mathbf{a}_x e^{-x^2}$, determine the time rate of change of the charge density at $(-1, 0, 0)$ and at $(1, 0, 0)$.
Solution. Using the continuity equation,

$$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \mathbf{J} = -\frac{\partial J_x}{\partial x} = 2xe^{-x^2}.$$

Thus at the point $(-1, 0, 0)$, $\partial \rho_v / \partial t = -2e^{-1} < 0$, whereas at $(1, 0, 0)$ $\partial \rho_v / \partial t = 2e^{-1} > 0$. The current flows to the right.

Let us now explore the electromagnetic energy propagation in a material medium. We assume that the medium is linear, homogeneous and isotropic as far as its electromagnetic properties are concerned,

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}, \tag{2.82}$$

and the current obeys Ohm's law

$$\mathbf{J} = \sigma \mathbf{E}. \tag{2.83}$$

In view of Eqs. (2.82) and (2.83), Maxwell's equations can be cast into the form

$$\nabla \cdot \mathbf{E} = \rho_v / \epsilon, \qquad \nabla \cdot \mathbf{H} = 0; \tag{2.84}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{2.85}$$

and

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{2.86}$$

Next, taking a dot product of Eq. (2.85) with $\mathbf{H}$ and Eq. (2.86) with $\mathbf{E}$, we obtain

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = -\frac{\mu}{2} \frac{\partial H^2}{\partial t}, \tag{2.87}$$

and

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \sigma E^2 + \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \sigma E^2 + \frac{\epsilon}{2} \frac{\partial E^2}{\partial t}. \tag{2.88}$$

Recalling the vector calculus identity,

$$\nabla \cdot (\mathbf{R} \times \mathbf{Q}) = \mathbf{Q} \cdot (\nabla \times \mathbf{R}) - \mathbf{R} \cdot (\nabla \times \mathbf{Q}), \tag{2.89}$$

for any vector fields $\mathbf{R}$ and $\mathbf{Q}$, and choosing $\mathbf{Q} = \mathbf{H}$ and $\mathbf{R} = \mathbf{E}$, and subtracting Eq. (2.88) from Eq. (2.87), we arrive at

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) - \sigma E^2. \tag{2.90}$$

Integrating Eq. (2.90) over the volume and using the divergence theorem on the l.h.s, we obtain

$$\oint_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t} \int_v dv \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) - \int_v dv \, \sigma E^2. \tag{2.91}$$

Finally, rearranging terms we obtain the electromagnetic energy conservation law in the form

$$\boxed{\frac{\partial}{\partial t}\int_v dv\, w_{\text{em}} = -\oint_S d\mathbf{S}\cdot\mathcal{P} - \int_v dv\sigma E^2}. \tag{2.92}$$

Here

$$\boxed{w_{\text{em}} = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2}, \tag{2.93}$$

is the **electromagnetic energy density** in J/m^3 and

$$\boxed{\boldsymbol{\mathcal{P}} = \mathbf{E}\times\mathbf{H}}, \tag{2.94}$$

is the so-called **Poynting vector** representing the **instantaneous electromagnetic power flow**, i.e., the electromagnetic power flowing per unit cross section in the medium, W/m^2. The second term on the r.h.s. of Eq. (2.92) describes ohmic losses. Thus, the electromagnetic energy conservation law asserts that the electromagnetic energy inside a finite volume can only change if the energy flows in or out of the volume through its surface and is lost inside to ohmic losses. Note the conservation law (2.92) is a direct consequence of Maxwell's equations.

Example 2.9. Given $\mathbf{B} = \mathbf{a}_y B_0 z\cos\omega t$ and its is known that E has only an x-component, determine the electric field generated by this magnetic field.

Solution. The Faraday law implies

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} = \mathbf{a}_y\omega B_0 z\sin\omega t. \tag{2.95}$$

As there is only an x-component of $\mathbf{E}$, $\mathbf{E} = \mathbf{a}_x E(x,y,z,t)$*, say, we have*

$$\nabla\times\mathbf{E} = \mathbf{a}_y\partial_z E - \mathbf{a}_z\partial_y E. \tag{2.96}$$

On comparing Eqs. (2.95) and (2.96), we conclude that

$$\partial_y E = 0, \qquad \partial_z E = \omega B_0 z\sin\omega t. \tag{2.97}$$

It follows that

$$E(x,z,t) = \frac{\omega B_0 z^2}{2}\sin\omega t + f(x,t).$$

Here $f(x,t)$ is an arbitrary function of time. In the limit $\omega = 0$, the magnetic field is static and hence it cannot generate any electric field. Therefore, we conclude that $f(x,t) = 0$. Thus, $\underline{\mathbf{E} = \mathbf{a}_x\omega B_0(z^2/2)\sin\omega t}$.

Chapter 3

Plane electromagnetic waves

3.1 Fundamentals of wave motion

Definition. A **monochromatic wave** is a **periodic** function of both space and time. A physical field is said to behave as a monochromatic wave if it oscillates sinusoidally in space and time. Consider, for simplicity, a one-dimensional scalar field $U(x,t)$–where x and t stand for the spatial coordinate and time–such as a density or pressure wave, for instance. If the field propagates to the right, say, as a monochromatic wave, one can write

$$U(x,t) = A\cos(\omega t - kx + \theta_0). \tag{3.1}$$

The latter can be rewritten in the phasor form as

$$U(x,t) = \mathrm{Re}\{U_0 e^{j(\omega t - kx)}\}. \tag{3.2}$$

where $U_0 = Ae^{j\theta_0}$ is a complex amplitude of the wave. Note that the field remains unchanged when translated in space by λ and in time by T. Thus, λ and T are the **wavelength** and **period** of the wave, defined by the expressions

$$k\lambda = 2\pi; \qquad \omega T = 2\pi, \tag{3.3}$$

implying that

$$\boxed{\lambda = \frac{2\pi}{k}}, \qquad \boxed{T = \frac{2\pi}{\omega}}. \tag{3.4}$$

In Eq. (3.4), k is a **wave number** and ω an **angular frequency** of the wave. The concepts of wavelength and period are illustrated in Fig. 3. 1.

The introduced angular frequency ω is measured in radians per second. Alternatively, a **linear frequency** ν–measured in hertz (Hz)–can be introduced; the former is related to the latter viz.,

$$\boxed{\omega = 2\pi\nu}. \tag{3.5}$$

Further, the quantity $\theta = \omega t - kx + \theta_0$ is called the **phase** of the wave and θ_0 the initial phase. The propagation velocity of a monochromatic wave can be inferred by

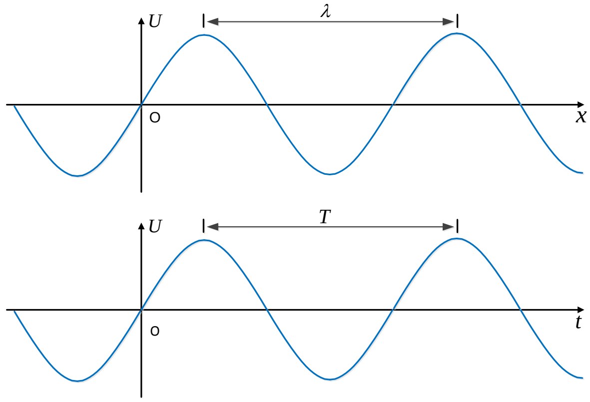

Figure 3.1: Illustrating the definitions of the wavelength and period of the wave.

following the movement of a fixed point on the **wave-front**, defined as

$$\omega t - kx + \theta_0 = const.$$

Assume we fix a point P on the wave-front as is indicated in Fig. 3. 2b. It follows that

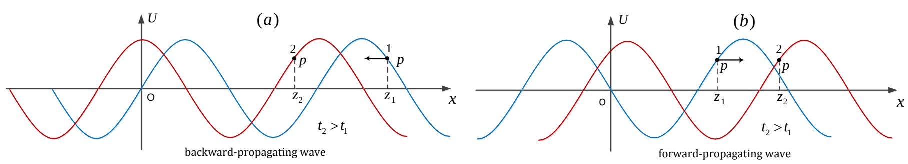

Figure 3.2: Forward-and backward-propagating plane waves.

the velocity of P can be found as

$$v_p = \frac{dx}{dt} = \frac{\omega}{k}. \tag{3.6}$$

The velocity defined by Eq. (3.6) is referred to as the **phase velocity** of the wave.

So far we have only studied the wave propagating to the right, the so-called forward-propagating wave. All the definitions are equally applicable to the backward-propagating waves, which can be expressed as

$$U(x,t) = A\cos(\omega t + kx + \theta_0) = \text{Re}\{U_0 e^{j(\omega t + kx)}\}. \tag{3.7}$$

The motion of a backward-propagating wave is sketched in Fig. 3.2a. Thus, one can express any one-dimensional monochromatic wave as

$$U_{\mp} = \text{Re}\{U_0 e^{j(\omega t \mp kx)}\}. \tag{3.8}$$

Note that on introducing the wave vector $\mathbf{k} = k\mathbf{a}_x$ and the position vector,

$$\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z,$$

one can rewrite, $kx = \mathbf{k} \cdot \mathbf{r}$. This identity hints to the generalization of our definition to a monochromatic wave, propagating in the direction, specified by the **wave vector**,

$$\mathbf{k} = k_x\mathbf{a}_x + k_y\mathbf{a}_y + k_z\mathbf{a}_z,$$

as

$$\boxed{U(\mathbf{r}, t) = \mathrm{Re}\{U_0 e^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\}}. \tag{3.9}$$

The wave defined by Eq. (3.9) is referred to as a **plane wave** because its wavefront–that is a surface of the constant phase–is a plane such that

$$\boxed{\mathbf{k} \cdot \mathbf{r} - \omega t = const}. \tag{3.10}$$

As a consequence of a finite wave speed, there is a time delay between an emitted and received wave signals which can be used in various applications.

Example 3. 1. A radar signal sent from the earth to the moon is received back on earth after a time delay of approximately 2.6 sec. Given the speed of light in vacuum, $c = 3 \times 10^8$ **m/s, estimate the distance between the earth surface and the moon.**

Solution. Assume the sought distance is R. *The time delay for the light round trip to the moon,* $\Delta t = 2R/c$. *It follows that* $\underline{R = c\Delta t/2 \simeq 3.9 \times 10^5}$ *km.*

Finally, we observe that monochromatic waves represent only a particular–albeit very important–class of waves. In general, a wave can contain many monochromatic components. In this case, the wave is called a **wave packet**. Any wave packet can be made up of a finite (infinite) number of monochromatic components via Fourier series (transform).

3.2 Doppler effect

Whenever there is a relative motion of a time-harmonic wave source and a receiver, the wave frequency detected by the latter differs from that emitted by the former. This phenomenon is called the **Doppler effect**. The Doppler effect is a purely kinematic effect and it takes place for waves of any physical nature whatsoever.

First, we present a "hand-waving" derivation of the Doppler effect for a source moving along a straight line toward a receiver at rest. The situation is illustrated in Fig. 3. 3. When the source is at rest, $u = 0$, the receiver detects $\nu_0 = v/\lambda_0$ wave crests per second, where v is a wave speed and λ_0 the wavelength of a monochromatic wave emitted by the source at rest. If, however, the source moves with the velocity $\mathbf{u} = u\mathbf{a}_x$, the wave speed in the receiver reference frame is $v_{eff} = v + u$. Consequently, the receiver detects $\nu = v_{eff}/\lambda_0 = (v + u)/\lambda_0 = \nu_0 + u/\lambda_0$ crests per second, implying that the wave frequency at the receiver shifts to

$$\boxed{\nu = \nu_0 + u/\lambda_0 = \nu_0(1 + u/v)}. \tag{3.11}$$

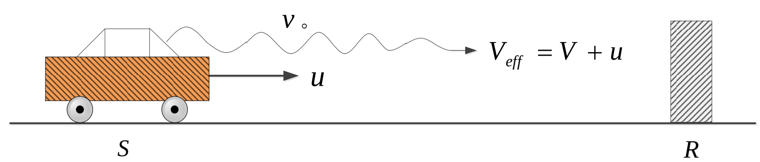

Figure 3.3: Doppler effect when a source S moves toward a receiver R.

Consider now a general plane monochromatic wave, represented in the receiver reference frame, where the source moves with the velocity $\mathbf{u}$ relative to the receiver, as

$$U(\mathbf{r},t) = \text{Re}\{U_0 e^{j(\mathbf{k}_0\cdot\mathbf{r}-\omega t)}\}. \tag{3.12}$$

The same wave can be represented in the source reference frame, where there is no relative motion, as

$$U(\mathbf{r}_0,t) = \text{Re}\{U_0 e^{j(\mathbf{k}_0\cdot\mathbf{r}_0-\omega_0 t)}\}. \tag{3.13}$$

In Eq. (3.12), $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t$ is a position vector of a point receiver R which, in the source frame, has the position vector $\mathbf{r}_0$, see Fig. 3.4.

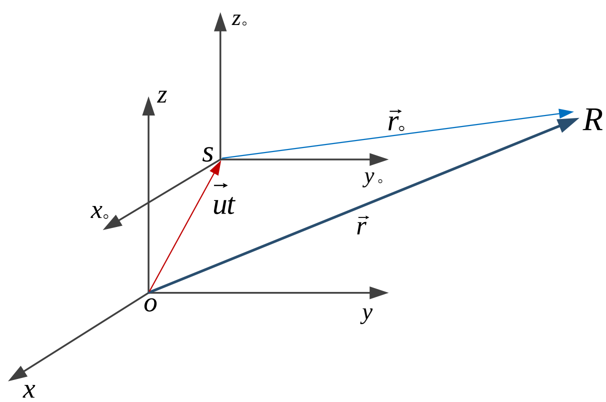

Figure 3.4: Illustrating the geometry for an arbitrary mutual motion of the source and receiver.

As this is actually the same wave, one can write

$$U(\mathbf{r},t) = U(\mathbf{r}_0,t). \tag{3.14}$$

It follows from Eqs. (3.13) upon some rearrangement in the exponent that

$$U(\mathbf{r},t) = \text{Re}\{U_0 e^{j(\mathbf{k}_0\cdot\mathbf{r}-\omega t)}\}, \tag{3.15}$$

where

$$\omega = \omega_0 + \mathbf{k}_0\cdot\mathbf{u} = \omega_0 + k_0 u\cos\theta = \omega_0\left(1+\frac{u\cos\theta}{v}\right), \tag{3.16}$$

is a general expression for a Doppler shifted angular frequency received at the receiver. In Eq. (3.16), v is the speed of the wave and θ is the angle the wave vector $\mathbf{k}_0$ makes with the source velocity $\mathbf{u}$. Alternatively, in terms of linear frequencies,

$$\boxed{\nu = \nu_0 \left(1 + \frac{u \cos\theta}{v}\right)}. \tag{3.17}$$

Notice that as $\cos\theta > 0$, implying that the source moves toward the receiver, the detected frequency at the receiver is greater, and if $\cos\theta < 0$–the source moves away from the receiver–it is less than the wave frequency of the source at rest. It is then sometimes said in the context of electromagnetic waves that the light frequency of a moving away source shifts to the "red"–because the wavelength increases–while it suffers a "blue" shift for an approaching source.

Example 3.2. Show that the wavelength of a monochromatic light source shifts to the blue (red), if the source moves toward (away from) the receiver.

Solution. It follows from Eq. (3.17) with $\cos\theta = \pm 1$ *that* $\nu = \nu_0(1 \pm u/c)$*, where* c *is the speed of light. Recall the definition of the wavelength,* $\lambda = c/\nu$ *and* $\lambda_0 = c/\nu_0$*. Hence,* $\lambda = \lambda_0/(1 \pm u/c) \simeq \lambda_0(1 \mp u/c)$*, as* $u \ll c$ *in most practical situations. Thus,* $\Delta\lambda = \lambda - \lambda_0 \simeq \mp\lambda_0 u/c$*;* $\Delta\lambda < 0$ *if* $\theta = 0$ *(arriving source), and* $\Delta\lambda > 0$ *if* $\theta = \pi$ *(departing source).*

In the limiting case of $\theta = 0$, Eq. (3.17) reduces to our "intuitive" result, Eq. (3.11). Finally, if $\theta = \pi/2$, i. e., the source moves orthogonally to the receiver, our general expression (3.17) predicts no frequency shift altogether.

3.3 Plane electromagnetic waves in free space

In the absence of charges and currents, Maxwell's equations in free space take the form

$$\nabla \cdot \mathbf{E} = 0, \tag{3.18}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{3.19}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{3.20}$$

and

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{3.21}$$

Building on our discussion of plane waves of any nature, we look for plane-wave solutions to the Maxwell equations in the form

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{E}_0 e^{j(\mathbf{k}\cdot\mathbf{r} - \omega t)}\}, \qquad \mathbf{H}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{H}_0 e^{j(\mathbf{k}\cdot\mathbf{r} - \omega t)}\}. \tag{3.22}$$

By linearity of Maxwell's equations in free space, we can drop the real part and deal with complex phasors describing the waves directly. The real part can be taken at the end of all calculations to yield physical (real) electric and magnetic fields of a plane wave.

To proceed, we require the following relations.

Example 3. 3. Show that for a plane wave given by Eq. (3.22), $\nabla \cdot \mathbf{E} = j\mathbf{k} \cdot \mathbf{E}$ and $\nabla \times \mathbf{E} = j\mathbf{k} \times \mathbf{E}$.

Solutions. In the Cartesian coordinates,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \left(E_{0x} \frac{\partial}{\partial x} + E_{0y} \frac{\partial}{\partial y} + E_{0z} \frac{\partial}{\partial z} \right) e^{j(k_x x + k_y y + k_z z)} e^{-j\omega t} \\ &= j(k_x E_{0x} + k_y E_{0y} + k_z E_{0z}) e^{j(\mathbf{k}\cdot\mathbf{r} - \omega t)} = j\mathbf{k} \cdot \mathbf{E}. \end{aligned} \tag{3.23}$$

The second relation is proven by analogy using the Cartesian coordinate representation of the curl.

The Maxwell equations in the plane-wave form can then be rewritten as

$$\boxed{\mathbf{k} \cdot \mathbf{E}_0 = 0}, \tag{3.24}$$

$$\boxed{\mathbf{k} \cdot \mathbf{H}_0 = 0}, \tag{3.25}$$

$$\boxed{\mathbf{k} \times \mathbf{E}_0 = \omega \mu_0 \mathbf{H}_0}, \tag{3.26}$$

and

$$\boxed{\mathbf{k} \times \mathbf{H}_0 = -\omega \epsilon_0 \mathbf{E}_0}. \tag{3.27}$$

In Eqs. (3.24) – (3.27) we dropped plane-wave phasors on both sides.

Next, we can exclude the magnetic field from the fourth Maxwell equation leading to

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) = -\epsilon_0 \mu_0 \omega^2 \mathbf{E}_0. \tag{3.28}$$

Using the "bac-cab" rule on the l.h.s of Eq. (3.28), we arrive at

$$\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_0) - k^2 \mathbf{E}_0 = -\epsilon_0 \mu_0 \omega^2 \mathbf{E}_0. \tag{3.29}$$

With the aid of Eq. (3.24), we obtain

$$(k^2 - \mu_0 \epsilon_0 \omega^2) \mathbf{E}_0 = 0, \tag{3.30}$$

implying that

$$\boxed{k = \omega \sqrt{\epsilon_0 \mu_0} = \omega / c} \tag{3.31}$$

where we introduced the speed of light in vacuum

$$\boxed{c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8} \text{ m/s}. \tag{3.32}$$

Equation (3.31) is a dispersion relation for plane electromagnetic waves in free space; it relates the wave number to the wave frequency. The complex amplitudes $\mathbf{E}_0$ and $\mathbf{H}_0$–which determine the directions of $\mathbf{E}$ and $\mathbf{H}$–are not independent, but are related by the Maxwell equations (3.26) or (3.27). For instance, from the knowledge of $\mathbf{E}_0$ one can determine $\mathbf{H}_0$ using Eq. (3.26),

$$\boxed{\mathbf{H}_0 = \frac{(\mathbf{a}_k \times \mathbf{E}_0)}{\eta_0}}, \tag{3.33}$$

where $\mathbf{a}_k = \mathbf{k}/k$ and η_0 is the **free space impedance** defined as

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 377\,\Omega. \tag{3.34}$$

By the same token, $\mathbf{E}_0$ can be inferred from $\mathbf{H}_0$ with the help of Eq. (3.27):

$$\boxed{\mathbf{E}_0 = -\eta_0(\mathbf{a}_k \times \mathbf{H}_0)}. \tag{3.35}$$

Example 3. 4. Show that E_0, k and H_0 are mutually orthogonal for a plane wave in free space.

Solution. It follows at once from the Maxwell equations, Eq. (3.24) and (3.25) that $\mathbf{E}_0 \perp \mathbf{k}$ *and* $\mathbf{H}_0 \perp \mathbf{k}$. *Taking a dot product of Eq. (3.26), say, with* $\mathbf{E}_0$ *we obtain,* $\mathbf{E}_0 \cdot (\mathbf{k} \times \mathbf{E}_0) = (\mathbf{E}_0 \times \mathbf{E}_0) \cdot \mathbf{k} = 0 = \omega\mu_0(\mathbf{E}_0 \cdot \mathbf{H}_0)$. *It follows that* $\mathbf{E}_0 \cdot \mathbf{H}_0 = 0$. *Thus,* $\mathbf{E}_0 \perp \mathbf{H}_0 \perp \mathbf{k}$. *See Fig. 3.5.*

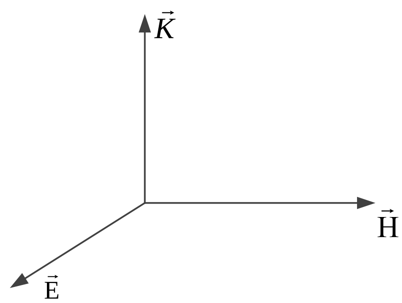

Figure 3.5: Mutual orientation of $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{k}$ of a plane wave propagating in free space.

Definition. The time evolution of the electric field vector is called **polarization**. Let us consider a plane wave propagating along the z-axis in free space. As, $\mathbf{k} = k\mathbf{a}_z$, and $\mathbf{E} \perp \mathbf{k}$, the electric field in the phasor form reads

$$\mathbf{E}(z,t) = \mathrm{Re}\{(\mathbf{a}_x|E_{0x}|e^{j\phi_{0x}} + \mathbf{a}_y|E_{0y}|e^{j\phi_{0y}})e^{j(kz-\omega t)}\}, \tag{3.36}$$

We will now show that, in general, the tip of the electric field vector moves around an ellipse as the time evolves. This general polarization is called **elliptic**. To proceed, we rewrite the complex amplitude in the rectangular form as

$$E_{0x}\mathbf{a}_x + E_{0y}\mathbf{a}_y = \underbrace{(\mathbf{a}_x|E_{0x}|\cos\phi_{0x} + \mathbf{a}_y|E_{0y}|\cos\phi_{0y})}_{\mathbf{U}} + j\underbrace{(\mathbf{a}_x|E_{0x}|\sin\phi_{0x} + \mathbf{a}_y|E_{0y}|\sin\phi_{0y})}_{\mathbf{V}}. \quad (3.37)$$

Note that $\mathbf{U}$ and $\mathbf{V}$ are not orthogonal which makes the situation tricky. We can however introduce a transformation from $\mathbf{U}$ and $\mathbf{V}$ to $\mathbf{u}$, $\mathbf{v}$ involving an auxiliary parameter θ such that

$$\mathbf{U} + j\mathbf{V} = (\mathbf{u} + j\mathbf{v})e^{j\theta}, \quad (3.38)$$

It follows at once from Eq. (3.38) that

$$\mathbf{U} = \mathbf{u}\cos\theta - \mathbf{v}\sin\theta, \qquad \mathbf{V} = \mathbf{u}\sin\theta + \mathbf{v}\cos\theta. \quad (3.39)$$

Inverting Eqs. (3.39), we obtain

$$\mathbf{u} = \mathbf{U}\cos\theta + \mathbf{V}\sin\theta, \qquad \mathbf{v} = \mathbf{U}\sin\theta - \mathbf{V}\cos\theta. \quad (3.40)$$

We can now use our freedom to choose θ wisely. In particular, choosing it such that $\mathbf{u}\cdot\mathbf{v} = 0$ (orthogonal axes), we obtain by taking the dot product of $\mathbf{u}$ and $\mathbf{v}$,

$$\tan 2\theta = \frac{2\mathbf{U}\cdot\mathbf{V}}{U^2 - V^2} \implies \theta = \frac{1}{2}\tan^{-1}\left(\frac{2\mathbf{U}\cdot\mathbf{V}}{U^2 - V^2}\right). \quad (3.41)$$

Here we made use of the trigonometric identities, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$. By combining Eqs. (3.37) and (3.38), we can rewrite our field as

$$\mathbf{E}(z,t) = \mathrm{Re}\{(\mathbf{u} + j\mathbf{v})e^{j(kz-\omega t+\theta)}\}. \quad (3.42)$$

Using the orthogonality of $\mathbf{u}$ and $\mathbf{v}$, we can write the two orthogonal components of the field, E_u and E_v as

$$E_u = u\cos(kz - \omega t + \theta), \qquad E_v = v\sin(kz - \omega t + \theta). \quad (3.43)$$

It follows from Eq. (3.43) that

$$\frac{E_u^2}{u^2} + \frac{E_v^2}{v^2} = 1, \quad (3.44)$$

where u and v are given by Eq. (3.40) and θ by Eq. (3.41). Eq. (3.44) manifestly represents an ellipse with the semi-major axis making the angle θ with the x-axis as is shown in Fig. 3.6. The tip of $\mathbf{E}$ can move either clockwise or counterclockwise along the ellipse; depending on the direction of motion of $\mathbf{E}$, the polarization is left-hand or right-hand elliptical. In the left-hand (right-hand) elliptical polarization, the fingers of your left (right) hand follow the direction of rotation and the thumb points to the wave propagation direction. Thus, for a general **elliptic** polarization, the electric field amplitude takes the form

$$\boxed{\mathbf{E}(z,t) = \mathbf{a}_x|E_{0x}|\cos(kz - \omega t + \phi_{0x}) + \mathbf{a}_y|E_{0y}|\cos(kz - \omega t + \phi_{0y})}. \quad (3.45)$$

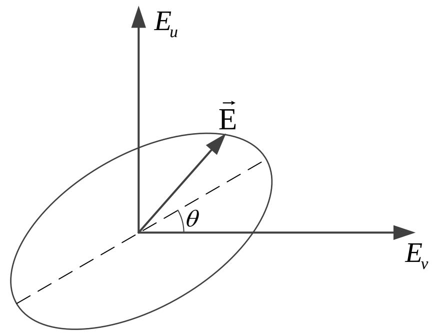

Figure 3.6: Illustrating elliptic polarization.

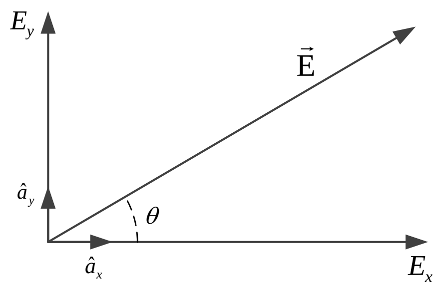

Figure 3.7: Illustrating plane polarization.

Although, in general, the electric field is elliptically polarized, there are two important particular cases.
Definition. The electric field is said to be **linearly polarized** if the phases of two orthogonal components of the field in Eq. (3.36) are the same, $\phi_{0x} = \phi_{0y}$.
In this case,

$$\boxed{\mathbf{E}(z,t) = (\mathbf{a}_x|E_{0x}| + \mathbf{a}_y|E_{0y}|)\cos(kz - \omega t + \phi_0)}, \tag{3.46}$$

and the electric field is always directed along the line making the angle

$$\alpha = \tan^{-1}(E_{0y}/E_{0x}) \tag{3.47}$$

with the x-axis as is shown in Fig. 3.7.
Definition. If the phases of the two orthogonal components in Eq. (3.37) differ by $\pi/2$, and $|E_{0x}| = |E_{0y}|$, the wave is said to be **circularly** polarized.
In this case

$$\boxed{\mathbf{E}(z,t) = |E_0|[\mathbf{a}_x\cos(kz - \omega t + \phi_0) \mp \mathbf{a}_y\sin(kz - \omega t + \phi_0)]}. \tag{3.48}$$

In a circularly polarized wave, the $\mathbf{E}$ has the same magnitude but is moving along

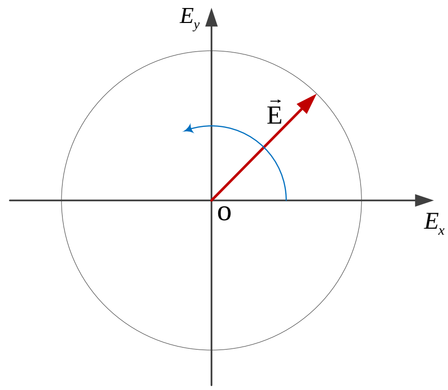

Figure 3.8: Illustrating circular polarization.

the circle. In the case of “-” sign in Eq. (3.48), $\mathbf{E}$ moves counterclockwise around the circle and the wave is **left circularly** polarized; for the “+” sign it is **right circularly** polarized.

Example 3. 5. Determine the polarization of the electromagnetic wave $\mathbf{E}(x,t) = (\mathbf{a}_y A - \mathbf{a}_z B)\sin(kx - \omega t)$.

Solution. The wave is linearly polarized at the angle $\theta = \tan^{-1}(A/B)$ with the z-axis; it propagates in the positive x-direction.

Consider the power flow associated with the plane wave, specified by the Poynting vector, $\boldsymbol{\mathcal{P}} = \mathbf{E}\times\mathbf{H}$. In general, both fields oscillate with rather high frequencies such that a more sensible–and actually detectable–measure of the power flow is the time-averaged Poynting vector, defined as

$$\langle\boldsymbol{\mathcal{P}}(\mathbf{r})\rangle = \frac{1}{T}\int_0^T dt\boldsymbol{\mathcal{P}}(\mathbf{r},t), \tag{3.49}$$

where T is the wave period. Using the fact that for any complex number, $\text{Re}(z) = (z+z^*)/2$, we can rewriting Eq. (3.22) as

$$\mathbf{E} = \frac{1}{2}\left[\mathbf{E}_0 e^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E}_0^* e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right];\ \ \mathbf{H} = \frac{1}{2}\left[\mathbf{H}_0 e^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{H}_0^* e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right], \tag{3.50}$$

We can obtain for the instantaneous Poynting vector the expression

$$\begin{aligned}\boldsymbol{\mathcal{P}} \;\; &= \;\; \frac{1}{4}\left(\mathbf{E}_0\times\mathbf{H}_0^* + \mathbf{E}_0^*\times\mathbf{H}_0\right) \\ &+ \frac{1}{2}\left[(\mathbf{E}_0\times\mathbf{H}_0)e^{2j(\mathbf{k}\cdot\mathbf{r}-\omega t)} + (\mathbf{E}_0^*\times\mathbf{H}_0^*)e^{-2j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right].\end{aligned} \tag{3.51}$$

It follows from Eqs. (3.49) and (3.51) that the last two terms on the r.h.s. of Eq. (3.51) average to zero as complex exponentials oscillate at twice the frequency and are averaged over a full period of the wave. As a result, the average Poynting vector of any plane wave, regardless of its polarization, takes the form

$$\boxed{\langle \boldsymbol{\mathcal{P}} \rangle = \frac{1}{2}\mathrm{Re}(\mathbf{E}_0 \times \mathbf{H}_0^*)}. \tag{3.52}$$

Example 3. 5. Determine the average Poynting vector of a plane wave $\mathbf{H} = (10\mathbf{a}_y - 20\mathbf{a}_z)\sin(\omega t - 40x)$ A/m propagating in free space.
Solution. In free space, $\omega = kc = 40\times 3\times 10^8 = 12\times 10^9$ rad/s. The magnetic field can be written in the phasor form as $\mathbf{H} = Re\{(10\mathbf{a}_y - 20\mathbf{a}_z)e^{j(\pi/2+kx-\omega t)}\}$, implying that the plane wave is linearly polarized at the angle $\theta = -\tan^{-1}(2)$ to the y-axis propagating along the x-axis, $\mathbf{a}_k = \mathbf{a}_x$. Here $\mathbf{H}_0 = (10\mathbf{a}_y - 20\mathbf{a}_z)e^{j\pi/2}$. Using Eq. (3.35), we obtain $\mathbf{E}_0 = -\eta_0[\mathbf{a}_x \times (10\mathbf{a}_y - 20\mathbf{a}_z)]e^{j\pi/2} = -\eta_0(10\mathbf{a}_z + 20\mathbf{a}_y)e^{j\pi/2}$. Note that $\mathbf{E}_0 \cdot \mathbf{H}_0 = 0$ as should be. Thus, $\langle \boldsymbol{\mathcal{P}} \rangle = -\eta_0[(10\mathbf{a}_z + 20\mathbf{a}_y) \times (10\mathbf{a}_y - 20\mathbf{a}_z)]/2 = \underline{0.25\eta_0\mathbf{a}_x \simeq 94.25\mathbf{a}_x\, kW/m^2}$

Example 3.6. Show that the instantaneous Poynting vector of a circularly polarized plane wave in free space is independent of either time or the propagation distance.
Solution. For a circularly polarized plane wave, propagating in the positive z-direction, say,

$$\mathbf{E} = |E_0|[\mathbf{a}_x \cos(kz-\omega t+\phi_0) \pm \mathbf{a}_y \sin(kz-\omega t+\phi_0)] = Re\left[E_0(\mathbf{a}_x \mp j\mathbf{a}_y)e^{j(kz-\omega t)}\right], \tag{3.53}$$

implying that

$$\mathbf{E}_0 = E_0(\mathbf{a}_x \mp j\mathbf{a}_y). \tag{3.54}$$

Applying Eq. (3.33), with Eq. (3.54), we obtain

$$\mathbf{H}_0 = \frac{(\mathbf{a}_z \times \mathbf{E}_0)}{\eta_0} = \frac{E_0}{\eta_0}(\mathbf{a}_y \pm j\mathbf{a}_x).$$

It follows that

$$\begin{aligned}\mathbf{H} &= Re\left[\frac{E_0}{\eta_0}(\mathbf{a}_y \pm j\mathbf{a}_x)e^{j(kz-\omega t)}\right] \\ &= \frac{|E_0|}{\eta_0}[\mathbf{a}_y \cos(kz - \omega t + \phi_0) \mp \mathbf{a}_x \sin(kz - \omega t + \phi_0)].\end{aligned} \tag{3.55}$$

Using Eqs. (3.53) and (3.55), we obtain

$$\begin{aligned}\boldsymbol{\mathcal{P}} &= \mathbf{E} \times \mathbf{H} = \frac{|E_0|^2}{\eta_0}[(\mathbf{a}_x \times \mathbf{a}_y)\cos^2(kz - \omega t + \phi_0) \\ &\quad - (\mathbf{a}_y \times \mathbf{a}_x)\sin^2(kz - \omega t + \phi_0)] = \underline{\mathbf{a}_z\frac{|E_0|^2}{\eta_0}}.\end{aligned} \tag{3.56}$$

3.4 Plane waves in lossy media

We begin by making two assumptions: First, we assume that neither ϵ nor σ is frequency dependent, and second, we assume that electromagnetic wave absorption comes only from the ohmic losses, implying that ϵ is purely real. The constitutive relations imply then

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}, \tag{3.57}$$

and

$$\mathbf{J} = \sigma\mathbf{E}. \tag{3.58}$$

With these two assumptions and constitutive relations, the Maxwell equations in the absence of free charges, $\rho_v = 0$, state

$$\nabla \cdot \mathbf{E} = 0, \tag{3.59}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{3.60}$$

$$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}, \tag{3.61}$$

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t}. \tag{3.62}$$

We seek a solution to Eqs. (3.59) through (3.62) in the form of an **inhomogeneous** plane wave propagating in the positive z-direction

$$\mathbf{E}(z,t) = \mathrm{Re}\{\mathbf{E}_0 e^{-\gamma z + j\omega t)}\}, \qquad \mathbf{H}(z,t) = \mathrm{Re}\{\mathbf{H}_0 e^{-\gamma z + j\omega t)}\}. \tag{3.63}$$

Here γ is, in general, a **complex propagation constant** with the imaginary part describing the amplitude attenuation of the wave as we shall see. Hence the name inhomogeneous plane wave. We can now introduce the complex wave vector $\mathbf{\Gamma}$ such that

$$\mathbf{\Gamma} = \mathbf{a}_z\gamma, \qquad \gamma z = \mathbf{\Gamma}\cdot\mathbf{r}. \tag{3.64}$$

On substituting from Eq. (3.63) into Eqs. (3.59) through (3.62), and using Eq. (3.64), together with the properties established in Exercise 3.3, we obtain

$$\mathbf{a}_z \cdot \mathbf{E}_0 = 0; \qquad \mathbf{a}_z \cdot \mathbf{H}_0 = 0, \tag{3.65}$$

and

$$\gamma(\mathbf{a}_z \times \mathbf{E}_0) = j\mu\omega\mathbf{H}_0, \tag{3.66}$$

$$-\gamma(\mathbf{a}_z \times \mathbf{H}_0) = \sigma\mathbf{E}_0 + j\epsilon\omega\mathbf{E}_0 = j\omega\left(\epsilon - \frac{j\sigma}{\omega}\right)\mathbf{E}_0. \tag{3.67}$$

Eliminating $\mathbf{H}_0$ from Eqs. (3.66) and (3.67) we obtain

$$\gamma^2[\mathbf{a}_z \times (\mathbf{a}_z \times \mathbf{E}_0)] = \mu\epsilon_{\mathrm{eff}}\omega^2\mathbf{E}_0. \tag{3.68}$$

Here we introduced the notation

$$\epsilon_{\mathrm{eff}} = \epsilon - \frac{j\sigma}{\omega}. \tag{3.69}$$

With the help of Eq. (3.65) by analogy with the free space case, we arrive at the dispersion relation for the electromagnetic waves in lossy media

$$\gamma^2 = -\mu\omega^2\epsilon_{\text{eff}} = -\omega^2\epsilon\mu\left(1 - \frac{j\sigma}{\epsilon\omega}\right). \tag{3.70}$$

The latter can be represented as

$$\boxed{\gamma = (\alpha + j\beta)}, \tag{3.71}$$

where

$$\boxed{\beta = \omega\sqrt{\frac{\epsilon\mu}{2}\left(\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} + 1\right)}}, \tag{3.72}$$

and

$$\boxed{\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}\left(\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} - 1\right)}}, \tag{3.73}$$

as you worked out in your assignment 1.

Further, the electric and magnetic field amplitudes are related as

$$\mathbf{E}_0 = -\eta(\mathbf{a}_z \times \mathbf{H}_0), \tag{3.74}$$

and

$$\mathbf{H}_0 = \frac{(\mathbf{a}_z \times \mathbf{E}_0)}{\eta}, \tag{3.75}$$

where η is a complex impedance of the lossy medium, defined as

$$\eta = \sqrt{\frac{\mu/\epsilon}{1 - \frac{j\sigma}{\epsilon\omega}}}. \tag{3.76}$$

The latter can be written in the polar form–see the solution to Example 1. 6–as

$$\boxed{\eta = |\eta|e^{j\theta_\eta}}, \tag{3.77}$$

where

$$\boxed{|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\epsilon\omega}}. \tag{3.78}$$

Consider now a particular case of a linearly polarized in the x-direction plane wave which propagates in the positive z-direction. In this case in view of Eqs. (3.63), (3.71), (3.77), and (3.75), we obtain for the electric and magnetic fields

$$\boxed{\mathbf{E}(z,t) = \mathbf{a}_x E_0 e^{-\alpha z}\cos(\beta z - \omega t)}, \tag{3.79}$$

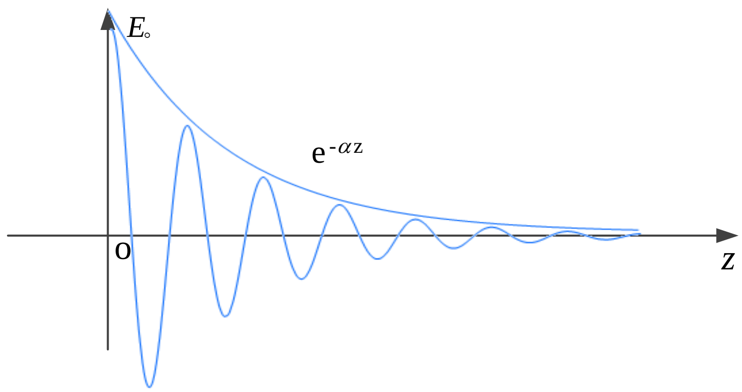

Figure 3.9: Inhomogeneous plane wave propagating in a lossy medium.

and

$$\boxed{\mathbf{H}(z,t) = \mathbf{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\beta z - \omega t - \theta_\eta)}. \tag{3.80}$$

It is seen from Eqs. (3.87) and 3.88) that in a lossy medium,

- The electric and magnetic field are harmonic waves with exponentially decaying amplitudes;
- The magnetic field lags behind the electric field in phase by θ_η.

The field attenuation is measured in nepers per meter (Np/m); and attenuation of 1Np implies the field amplitude is reduced e times. The wave power loss of 1 neper can be expressed in decibels as

$$1\,\mathrm{Np} = 20\log_{10} e = 8.686\,\mathrm{dB}.$$

Next, let us evaluate the magnitude ratio of conducting and displacement currents generated by the wave on propagation in the medium

$$\mathbf{J}_c = \sigma \mathbf{E}; \qquad \mathbf{J}_d = \epsilon \frac{\partial \mathbf{E}}{\partial t} = j\epsilon\omega \mathbf{E}. \tag{3.81}$$

Thus,

$$\boxed{\frac{|\mathbf{J}_c|}{|\mathbf{J}_d|} = \frac{\sigma}{\epsilon\omega} \equiv \tan\theta}. \tag{3.82}$$

Eq. (3.82) defines the so-called **loss tangent**. Note that the loss angle θ is related to the aforementioned phase lag angle θ_η viz.,

$$\theta = 2\theta_\eta. \tag{3.83}$$

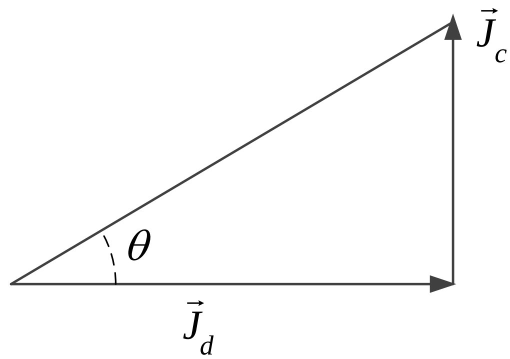

Figure 3.10: Illustrating the loss angle concept.

Depending on the size of the loss angle, two interesting cases emerge.
Lossless dielectrics, $\sigma = 0$. In this case, one can set σ equal to zero in all the above expressions. We arrive at

$$\boxed{\mathbf{E}_0 = -\eta(\mathbf{a}_k \times \mathbf{H}_0)}, \tag{3.84}$$

and

$$\boxed{\mathbf{H}_0 = \frac{(\mathbf{a}_k \times \mathbf{E}_0)}{\eta}}, \tag{3.85}$$

where

$$\boxed{\beta = \omega\sqrt{\epsilon\mu}}, \qquad \boxed{\eta = \sqrt{\mu/\epsilon}}. \tag{3.86}$$

In particular, for a linearly polarized plane wave propagating in the z-direction, we obtain

$$\mathbf{E}(z,t) = \mathbf{a}_x E_0 \cos(\beta z - \omega t), \tag{3.87}$$

and

$$\mathbf{H}(z,t) = \mathbf{a}_y \frac{E_0}{|\eta|} \cos(\beta z - \omega t). \tag{3.88}$$

The wave is simply a plane wave of the wavelength

$$\boxed{\lambda = 2\pi/\beta}, \tag{3.89}$$

propagating with no loss and phase velocity

$$\boxed{v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{\mu_r \epsilon_r}}}. \tag{3.90}$$

Good conductors, $\sigma/\epsilon\omega \gg 1$. In this case, we can get approximately,

$$\alpha = \beta = \sqrt{\frac{\mu\sigma\omega}{2}} = \frac{1}{\delta}, \tag{3.91}$$

and

$$\boxed{\eta = \frac{(1+j)}{\delta\sigma}}, \qquad \boxed{\theta_\eta = \pi/4}. \tag{3.92}$$

Here we introduced the **skin depth** by the expression

$$\boxed{\delta = \sqrt{\frac{2}{\mu\sigma\omega}}}. \tag{3.93}$$

Thus, for a linearly polarized plane wave along the x-axis, say, we get

$$\mathbf{E}(z,t) = \mathbf{a}_x E_0 e^{-z/\delta}\cos(\beta z - \omega t), \tag{3.94}$$

and

$$\mathbf{H}(z,t) = \mathbf{a}_y \frac{E_0}{\eta} e^{-z/\delta}\cos(\beta z - \omega t - \pi/4). \tag{3.95}$$

For good conductors, $\sigma \to \infty$, the skin depth is quite small. At the microwave frequencies, for instance, δ ranges from 10^{-4} mm to 10^{-2} mm; the fields do not penetrate much into good conductors at the microwave or higher frequencies.

Example 3. 7. In a lossless nonmagnetic medium with $\epsilon_r = 4$, a uniform plane wave, $\mathbf{E} = 8\cos(\omega t - \beta z)\mathbf{a}_x - 6\mathbf{a}_y \sin(\omega t - \beta z)$ V/m propagates with the frequency of 30 MHz. Determine β, v_p, λ, η and H.

Solution. $\beta = \omega\sqrt{\mu\epsilon} = \sqrt{\epsilon_r}\omega/c = 2\pi\nu\sqrt{\epsilon_r}/c = \underline{2\pi/5\,m/s}$. *It follows by definitions that* $\lambda = 2\pi/\beta = \underline{5\,m}$; $v_p = \omega/\beta = \underline{1.5\times 10^8\,m/s}$, *and* $\eta = \sqrt{\mu/\epsilon} = \eta_0/\sqrt{\epsilon_r} = \underline{\eta_0/2}$. *Next,* $\mathbf{E} = Re\{\mathbf{E}_0 e^{j(\omega t - \beta z)}\}$, *where* $\mathbf{E}_0 = 8\mathbf{a}_x + 6j\mathbf{a}_y$. *Thus,* $\mathbf{H}_0 = (\mathbf{a}_z \times \mathbf{E}_0)/\eta = (16/\eta_0)\mathbf{a}_y - (12j/\eta_0)\mathbf{a}_x$. *It follows that* $\mathbf{H} = Re\{\mathbf{H}_0 e^{j(\omega t - \beta z)}\} = \underline{(16/\eta_0)\cos(\omega t - \beta z)\mathbf{a}_y + (12/\eta_0)\sin(\omega t - \beta z)\mathbf{a}_x}$.

3.5 Group velocity

Let us consider a dispersive nonmagnetic medium with $\epsilon = \epsilon(\omega)$. We assume that all fields have a harmonic time dependence,

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[\mathbf{E}_\omega(\mathbf{r})e^{-j\omega t}] \qquad \mathbf{H}(\mathbf{r},t) = \mathrm{Re}[\mathbf{H}_\omega(\mathbf{r})e^{-j\omega t}]. \tag{3.96}$$

The constitutive relations at frequency ω take the form

$$\mathbf{D}_\omega = \epsilon(\omega)\mathbf{E}_\omega, \tag{3.97}$$

and

$$\mathbf{B}_\omega = \mu_0 \mathbf{H}_\omega. \tag{3.98}$$

Maxwell's equations for the time-harmonic components are obtained from Maxwell's equations in the space-time representation with the replacement $\partial_t \to -j\omega$, resulting in

$$\nabla \cdot \mathbf{E}_\omega = \rho_\omega/\epsilon(\omega). \tag{3.99}$$

$$\nabla \cdot \mathbf{H}_\omega = 0, \tag{3.100}$$

$$\nabla \times \mathbf{E}_\omega = j\omega\mu_0 \mathbf{H}_\omega, \tag{3.101}$$

and

$$\nabla \times \mathbf{H}_\omega = \mathbf{J}_\omega - j\omega\epsilon(\omega)\mathbf{E}_\omega. \tag{3.102}$$

We now explore a generic linearly polarized wave packet evolution in a dispersive medium with no free charges and currents, $\rho_\omega = 0$ and $\mathbf{J}_\omega = 0$. Phasor Maxwell's equations reduce then to

$$\nabla \cdot \mathbf{E}_\omega = 0. \tag{3.103}$$

$$\nabla \cdot \mathbf{H}_\omega = 0, \tag{3.104}$$

$$\nabla \times \mathbf{E}_\omega = j\omega\mu_0 \mathbf{H}_\omega, \tag{3.105}$$

and

$$\nabla \times \mathbf{H}_\omega = -j\omega\epsilon(\omega)\mathbf{E}_\omega. \tag{3.106}$$

One can take the curl of Eq. (3.105) and use (3.106) to eliminate the magnetic field to yield

$$\nabla \times (\nabla \times \mathbf{E}_\omega) = \omega^2\mu_0\epsilon(\omega)\mathbf{E}_\omega. \tag{3.107}$$

We look for a solution to Eq. (3.107) describing a linearly polarized wave packet, propagating in the positive z-direction:

$$\mathbf{E}(z,t) = \mathbf{a}_x \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\omega\, \mathcal{E}(\beta, \omega - \omega_c)\, e^{j(\beta z - \omega t)}. \tag{3.108}$$

Here we introduced the carrier frequency ω_c and the envelope of the wave packet, $\mathcal{E}(\beta, \omega - \omega_c)$, centered around ω_c. On substituting from Eqs. (3.108) into Eq. (3.107), we obtain the relation

$$[\beta^2 - \omega^2\mu_0\epsilon(\omega)]\mathcal{E}(\beta, \omega - \omega_c) = 0. \tag{3.109}$$

Note that as β and ω are treated as independent variables, the only way Eq. (3.109) can be satisfied for all pairs of β and ω is if

$$\mathcal{E}(\beta, \omega - \omega_c) = E_0(\omega - \omega_c)\delta[\beta^2 - \omega^2\mu_0\epsilon(\omega)]. \tag{3.110}$$

Here δ is a Dirac delta function. We recall the following property of the δ function of an arbitrary argument $f(x)$

$$\delta[f(x)] = \sum_s \frac{1}{|f'(x_s)|}\delta(x - x_s), \tag{3.111}$$

where $\{x_s\}$ are the roots of the function $f(x)$, and the prime denotes a derivative. It follows from Eqs. (3.110) and (3.111) that

$$\mathcal{E}(\beta,\omega) = E_{0+}(\omega-\omega_c)\delta[\beta - \omega\sqrt{\mu_0\epsilon(\omega)}] + E_{0-}(\omega-\omega_c)\delta[\beta + \omega\sqrt{\mu_0\epsilon(\omega)}] \tag{3.112}$$

As we assume that the wave packet is forward-propagating,

$$E_{0-}(\omega - \omega_c) = 0. \tag{3.113}$$

The dispersion relation between β and ω follows from Eqs (3.112) and (3.113):

$$\beta(\omega) = \omega\sqrt{\mu_0\epsilon(\omega)}. \tag{3.114}$$

It follows at once from Eqs. (3.112), (3.113) and (3.108) that

$$\mathbf{E}(z,t) = \mathbf{a}_x \int_{-\infty}^{\infty} d\omega\, E_0(\omega - \omega_c) e^{j[\beta(\omega)z - \omega t]}. \tag{3.115}$$

Here the propagation constant satisfies the dispersion relation (3.114).

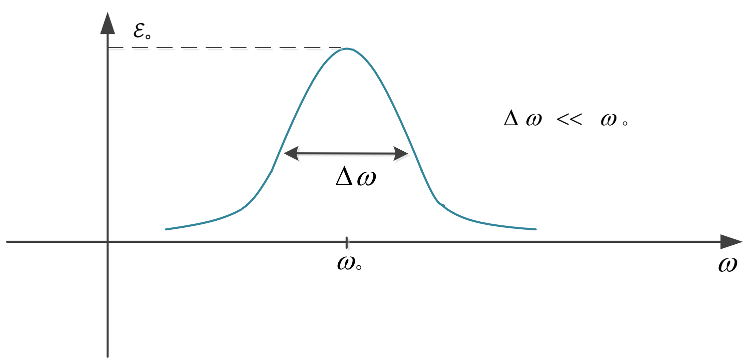

Figure 3.11: Narrow-band wave envelope.

Assume now that the wave packet has a carrier frequency ω_c far from any medium resonances and its bandwidth $\Delta\omega$ is sufficiently narrow such that $\Delta\omega \ll \omega_c$. In this approximation–referred to as the slowly-varying envelope approximation (SVEA)–a "fast" carrier modulates a "slow" envelope as sketched in Fig. 3.12. Under the circumstances, we can expand β in a Taylor series around ω_c and keep only first two terms of the expansion:

$$\beta(\omega) \simeq \beta_c + \beta_c'(\omega - \omega_c). \tag{3.116}$$

Here $\beta_c \equiv \beta(\omega_c)$ and $\beta_c' \equiv \beta'(\omega_c)$. Thus,

$$\mathbf{E}(z,t) = \mathbf{a}_x e^{j(\beta_c z - \omega_c t)} \int_{-\infty}^{\infty} d\omega\, E_0(\omega - \omega_c) e^{-j(\omega-\omega_c)(t-\beta_c' z)}. \tag{3.117}$$

and on introducing the new variable, $\omega' = \omega - \omega_c$ in Eq. (3.117), we obtain

$$\mathbf{E}(z,t) \simeq \mathbf{a}_x e^{j(\beta_c z - \omega_c t)} \int_{-\infty}^{\infty} d\omega'\, E_0(\omega') e^{-j\omega'(t-\beta_c' z)}. \tag{3.118}$$

It can be inferred from Eq. (3.118) using the shifting property of Fourier transforms

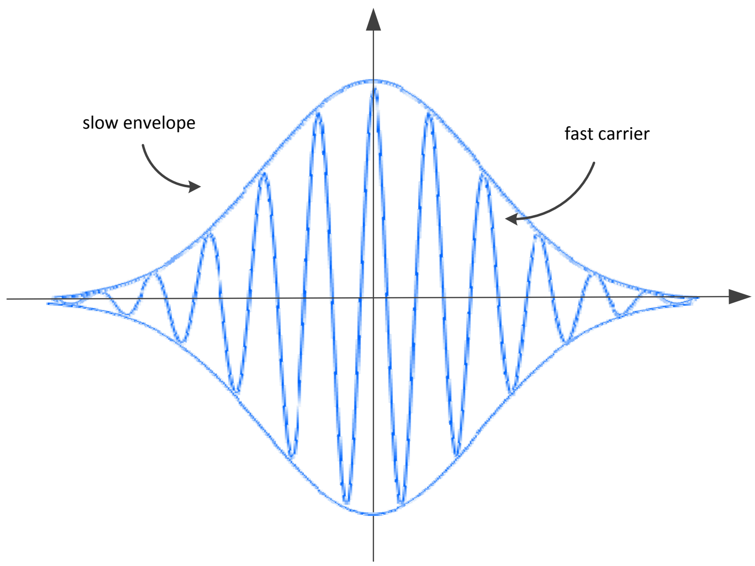

Figure 3.12: Fast carrier and the slow envelope modulated by the carrier.

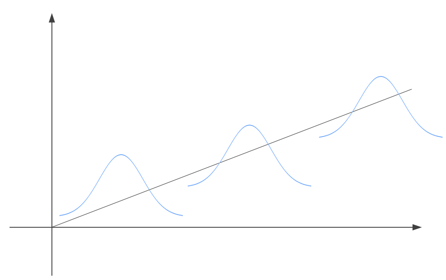

Figure 3.13: Distortionless propagation of the wave packet with the group velocity.

that at the adopted level of approximation, the wave packet profile remains unchanged; it simply propagates as a whole to the right with the speed v_g, i. e.,

$$\mathbf{E}(z,t) = \mathbf{E}(0, t - \beta_c' z) = \mathbf{E}(0, t - z/v_g). \tag{3.119}$$

Here the **group velocity** is introduced viz.,

$$\boxed{v_g = \frac{1}{\beta_c'} = \left.\frac{1}{\frac{d\beta(\omega)}{d\omega}}\right|_{\omega=\omega_c} = \left.\frac{d\omega}{d\beta}\right|_{\omega=\omega_c}}. \tag{3.120}$$

At the same time, the carrier propagates with the phase velocity

$$\boxed{v_p = \frac{\omega_c}{\beta(\omega_c)}}. \tag{3.121}$$

Note that we can define the phase velocity of any monochromatic component with the frequency ω within the envelope as

$$v_p = \frac{\omega}{\beta(\omega)}. \tag{3.122}$$

Note also that as long as the phase velocity is just a velocity of a phase of the wave it can take on any value, even greater than the speed of light.

Example 3. 8. The ionosphere of earth contains a number of free electrons such that it can be modeled as a nonmagnetic medium with a Drude-type permittivity, $\epsilon(\omega) = \epsilon_0(1 - \omega_p^2/\omega^2)$. Show that the phase velocity in the ionosphere exceeds the speed of light in vacuum.

Solution. By definition,

$$v_p = \frac{\omega}{\beta(\omega)} = \frac{1}{\sqrt{\mu_0 \epsilon(\omega)}} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}} \geq c.$$

Example 3. 9. Show that the group and phase velocities are related as

$$v_g = v_p + \beta \frac{dv_p}{d\beta},$$

or

$$v_g = \frac{v_p}{1 - \frac{\omega}{v_p}\frac{dv_p}{d\omega}}.$$

Solution. First, recall that $v_p = \omega/\beta$ implying that $\omega = \beta v_p$ or $\beta = \omega/v_p$. Eliminating ω, we obtain

$$v_g = \frac{d\omega}{d\beta} = \frac{d}{d\beta}(\beta v_p) = v_p + \beta \frac{dv_p}{d\beta}.$$

Alternatively, eliminating β, we get

$$v_g = \frac{1}{\frac{d\beta(\omega)}{d\omega}} = \frac{1}{v_p^{-1} - \omega v_p^{-2}\frac{dv_p}{d\omega}} = \frac{v_p}{1 - \frac{\omega}{v_p}\frac{dv_p}{d\omega}}. \tag{3.123}$$

Note that the group velocity is related to the envelope propagation. The envelope carries the wave packet energy and its propagation velocity can never exceed the speed of light in vacuum. It also follows from Eq. (3.123) that

- $\frac{dv_p}{d\omega} = 0 \Rightarrow v_p = v_g$, there is no dispersion;
- $\frac{dv_p}{d\omega} < 0 \Rightarrow v_g < v_p$, dispersion is **normal**;
- $\frac{dv_p}{d\omega} > 0 \Rightarrow v_g > v_p$, dispersion is **anomalous**.

3.6 Reflection of plane waves at normal incidence

We consider a plane electromagnetic wave propagating along the z-axis in a dielectric nonconducting medium with the constitutive parameters ϵ_1 and μ_1 (medium 1). The wave is impinged normally on the interface $z = 0$ separating the medium from a conducting medium characterized by ϵ_2, μ_2 and σ. The situation is schematically depicted in Fig. 3.14. As the incident wave is partially reflected back into medium 1 and partially transmitted into medium 2, there will be reflected and incident waves in medium 1 and a transmitted wave in medium 2.

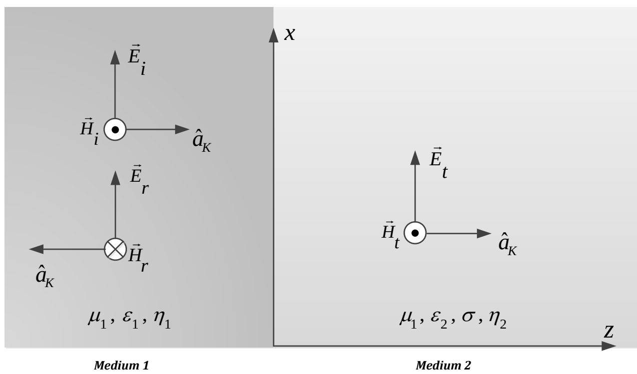

Figure 3.14: Normal incidence of a plane wave onto an interface separating transparent and lossy media.

We can now write down the fields in the media. Hereafter, we are going to write the fields in the complex form implying that the real part can be taken at the end of all calculations. First, there is an incident wave in medium 1:

$$\mathbf{E}_i(z,t) = \mathbf{E}_{0i} e^{j(k_i z - \omega_i t)}, \tag{3.124}$$

and

$$\mathbf{H}_i(z,t) = \mathbf{H}_{0i} e^{j(k_i z - \omega_i t)}, \tag{3.125}$$

where $\mathbf{E}_{0i}$ and $\mathbf{H}_{0i}$ are complex amplitudes of the electric and magnetic fields and β_i is the propagation constant. Assuming, for simplicity, that the incident field is linearly polarized along the x-axis, we get

$$\mathbf{E}_{0i} = E_{0i}\mathbf{a}_x, \tag{3.126}$$

and since the for incident wave $\mathbf{k}_i = k_i \mathbf{a}_z$, we obtain

$$\mathbf{H}_{0i} = \frac{(\mathbf{a}_z \times \mathbf{E}_{0i})}{\eta_i} = \mathbf{a}_y \frac{E_{0i}}{\eta_i}. \tag{3.127}$$

It can be inferred from Eqs. (3.124) through (3.127) that

$$\mathbf{E}_i(z,t) = \mathbf{a}_x E_{0i} e^{j(k_i z - \omega_i t)}, \tag{3.128}$$

and

$$\mathbf{H}_i(z,t) = \mathbf{a}_y \frac{E_{0i}}{\eta_i} e^{j(k_i z - \omega_i t)}. \tag{3.129}$$

By the same token, for the reflected wave,

$$\mathbf{E}_r(z,t) = \mathbf{a}_x E_{0r} e^{-j(k_r z + \omega_r t)}, \tag{3.130}$$

and

$$\mathbf{H}_r(z,t) = -\mathbf{a}_y \frac{E_{0r}}{\eta_r} e^{-j(k_r z + \omega_r t)}. \tag{3.131}$$

The minus sign in Eq. (3.131) is because the reflected wave propagates in the negative z-direction, $\mathbf{k}_r = -k_r \mathbf{a}_z$. Also we have to flip the direction of the magnetic field if we assume that the linearly polarized electric field does not change its polarization upon reflection and $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{k}$ form a right-hand system of mutually orthogonal vectors. The transmitted waves in medium 1 can be expressed in a similar fashion as

$$\mathbf{E}_t(z,t) = \mathbf{a}_x E_{0t} e^{j(\gamma_t z - \omega_t t)}, \tag{3.132}$$

and

$$\mathbf{H}_t(z,t) = \mathbf{a}_y \frac{E_{0t}}{\eta_t} e^{j(\gamma_t z - \omega_t t)}. \tag{3.133}$$

Next, the boundary conditions for the tangential field components at the interface $z = 0$ state

$$\mathbf{E}_{1\tau}|_{z=0} = \mathbf{E}_{2\tau}|_{z=0}, \qquad \mathbf{H}_{1\tau}|_{z=0} = \mathbf{H}_{2\tau}|_{z=0}. \tag{3.134}$$

Since Eq. (3.134) must be satisfied at any instant of time we stipulate that

$$e^{j(k_i z - \omega_i t)}|_{z=0} = e^{-j(k_r z + \omega_r t)}|_{z=0} = e^{j(\gamma_t z - \omega_t t)}|_{z=0}. \tag{3.135}$$

It follows at once from Eq. (3.135) that

$$\omega_i = \omega_r = \omega_t = \omega. \tag{3.136}$$

Eq. (3.133), in turn, leads to

$$k_i = k_r = k_1 = \omega\sqrt{\epsilon_1 \mu_1}, \tag{3.137}$$

and

$$\eta_i = \eta_r = \eta_1 = \sqrt{\mu_1/\epsilon_1}, \tag{3.138}$$

as well as

$$\gamma_t = \gamma_2 = \beta_2 + j\alpha_2, \tag{3.139}$$

and

$$\eta_t = \eta_2 = \sqrt{\frac{\mu_2/\epsilon_2}{1 - \frac{j\sigma}{\epsilon_2 \omega}}}. \tag{3.140}$$

The boundary conditions then imply

$$E_{0i} + E_{0r} = E_{0t} \tag{3.141}$$

and

$$\frac{E_{0i} - E_{0r}}{\eta_1} = \frac{E_{0t}}{\eta_2}. \tag{3.142}$$

Solving Eqs. (3.141) and (3.142), we obtain for the complex reflection and transmission coefficients, r and t, the equations

$$\boxed{r \equiv \frac{E_{0r}}{E_{0i}} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}}, \tag{3.143}$$

and

$$\boxed{t \equiv \frac{E_{0t}}{E_{0i}} = \frac{2\eta_2}{\eta_1 + \eta_2}}. \tag{3.144}$$

Note that

$$1 + r = t. \tag{3.145}$$

Example 3. 10. Show that the average Poynting vectors on both sides of the interface are equal.
Solution. Recall that $\langle \boldsymbol{\mathcal{P}} \rangle = \frac{1}{2} Re(\mathbf{E}_0 \times \mathbf{H}_0^*)$. *It follows that*

$$\begin{aligned} \langle \boldsymbol{\mathcal{P}}_1 \rangle &= \langle \boldsymbol{\mathcal{P}}_i \rangle + \langle \boldsymbol{\mathcal{P}}_r \rangle = \frac{1}{2}\left(\frac{|E_{0i}|^2}{\eta_1}\mathbf{a}_x \times \mathbf{a}_y\right) - \frac{1}{2}\left(r^2\frac{|E_{0i}|^2}{\eta_1}\mathbf{a}_x \times \mathbf{a}_y\right) \\ &= \frac{1}{2}(1 - r^2)\frac{|E_{0i}|^2}{\eta_1}\mathbf{a}_z = \frac{2\eta_2|E_{0i}|^2}{(\eta_1+\eta_2)^2}. \end{aligned} \tag{3.146}$$

By the same token,

$$\langle \boldsymbol{\mathcal{P}}_2 \rangle = \langle \boldsymbol{\mathcal{P}}_t \rangle = \frac{1}{2}\frac{t^2|E_{0i}|^2}{\eta_2}\mathbf{a}_z = \frac{2\eta_2|E_{0i}|^2}{(\eta_1+\eta_2)^2}, \tag{3.147}$$

Thus, the r.h.s of Eqs. (3.146) and (3.147) are equal which is a consequence of the energy conservation, of course.

Let us now examine important particular cases when medium 2 is a perfect dielectric, $\sigma = 0$, or a perfect conductor, $\sigma \to \infty$. In the first case, both impedances are real and there are transmitted and reflected homogeneous plane waves. In the second case, $\eta_2 \to 0$, implying that $E_{0t} = 0$, $H_{0t} = 0$ and $E_{0r} = -E_{0i}$. That is all power is reflected back into medium 1. This situation is illustrated in Fig. 3.15. Thus,

$$\mathbf{E}_i(z,t) = \mathbf{a}_x E_{0i} e^{j(k_1 z - \omega t)}, \tag{3.148}$$

$$\mathbf{H}_i(z,t) = \mathbf{a}_y \frac{E_{0i}}{\eta_i} e^{j(k_1 z - \omega t)}. \tag{3.149}$$

for the incident wave, and

$$\mathbf{E}_r(z,t) = -\mathbf{a}_x E_{0i} e^{-j(k_1 z + \omega t)}, \tag{3.150}$$

$$\mathbf{H}_r(z,t) = \mathbf{a}_y \frac{E_{0i}}{\eta_1} e^{-j(k_1 z + \omega t)}. \tag{3.151}$$

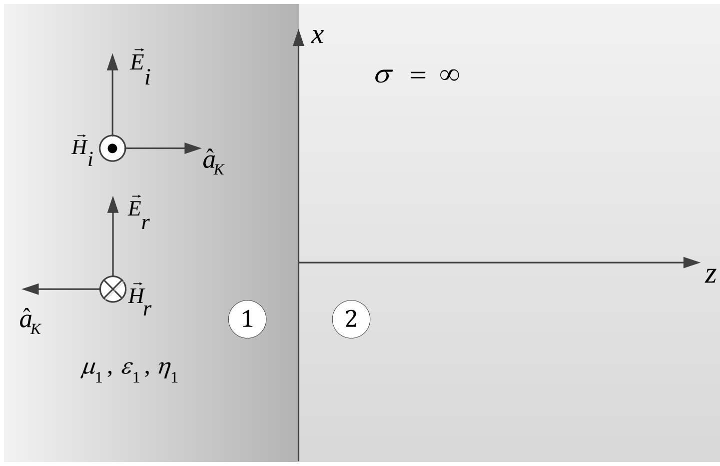

Figure 3.15: Normal incidence of a plane wave onto an interface separating a dielectric and a perfect conductor.

for the reflected one. The total electric and magnetic fields in medium 1 are then

$$\boxed{\mathbf{E}_1 = \text{Re}(\mathbf{E}_i + \mathbf{E}_r) = 2\mathbf{a}_x|E_{0i}|\sin k_1 z \sin\omega t}, \tag{3.152}$$

and

$$\boxed{\mathbf{H}_1 = \text{Re}(\mathbf{H}_i + \mathbf{H}_r) = 2\mathbf{a}_y\frac{|E_{0i}|}{\eta_1}\cos k_1 z \cos\omega t}. \tag{3.153}$$

These are standing waves because the wave does not travel, but simply oscillates separately in space and time.

Example 3.11. Show that standing waves do not transmit any power.

Solution. Let us work out the time-averaged power flow associated with a standing wave.

$$\langle \boldsymbol{\mathcal{P}} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \mathbf{a}_z \frac{|E_{0i}|^2}{\eta_1} \sin 2k_1 z \langle \sin 2\omega t \rangle = 0,$$

because $\sin 2\omega t$ *yields zero upon averaging over a wave period* $T = 2\pi/\omega$.

Example 3. 12. A right-hand circularly polarized wave, propagating in the positive z-direction is normally incident on a perfect conductor wall $z = 0$. Determine (a) the polarization of the reflected wave and (b) the induced current on the conducting wall.

Solution. (a) The incident wave can be written in the phasor form as $\mathbf{E}_i = Re[E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{j(kz-\omega t)}] = E_0[\cos(kz - \omega t)\mathbf{a}_x - \sin(kz - \omega t)\mathbf{a}_y]$. *The reflected wave is then* $\mathbf{E}_r = Re[E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j(kz+\omega t+\pi)}] = E_0[\cos(kz + \omega t + \pi)\mathbf{a}_x + \mathbf{a}_y \sin(kz + \omega t + \pi)\mathbf{a}_y]$. *Thus, the reflected wave is left-hand circularly polarized wave propagating in* *the negative z-direction. At the interface, dropping the harmonic dependence,*

$$\mathbf{H}_i = \frac{E_0}{\eta}\mathbf{a}_z \times (\mathbf{a}_x + j\mathbf{a}_y) = \frac{E_0}{\eta_1}(\mathbf{a}_y - j\mathbf{a}_x), \tag{3.154}$$

and

$$\mathbf{H}_r = \frac{E_0}{\eta}\mathbf{a}_z \times (\mathbf{a}_x + j\mathbf{a}_y) = \frac{E_0}{\eta_1}(\mathbf{a}_y - j\mathbf{a}_x). \tag{3.155}$$

The boundary conditions imply,

$$\mathbf{J}_s = -\mathbf{a}_z \times (\mathbf{H}_i + \mathbf{H}_r) = \underline{2\frac{E_0}{\eta_1}(\mathbf{a}_x - j\mathbf{a}_y)}. \tag{3.156}$$

3.7 Reflection of plane waves at oblique incidence

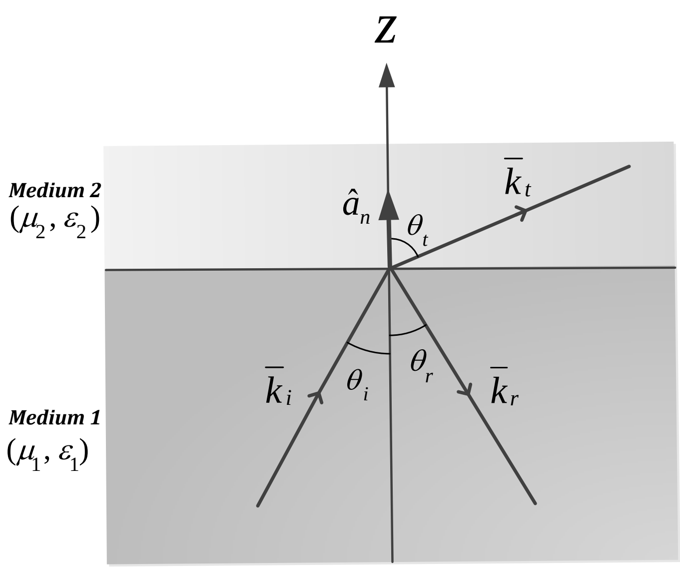

Figure 3.16: Illustrating Snell's law for oblique incidence of a plane wave.

We now explore refraction and reflection of plane electromagnetic waves at a flat interface of two lossless media. Here we assume that the interface is flat with its normal being along the z-axis. The boundary conditions at the flat interface $z = 0$ should hold at any point in the xy-plane and at any instant of time t, implying that

$$e^{j(\mathbf{k}_i\cdot\mathbf{r}-\omega_i t)}|_{z=0} = e^{j(\mathbf{k}_r\cdot\mathbf{r}-\omega_r t)}|_{z=0} = e^{j(\mathbf{k}_t\cdot\mathbf{r}-\omega_t t)}|_{z=0}. \tag{3.157}$$

It follows at once from Eq. (3.157) that

$$\omega_i = \omega_r = \omega_t = \omega. \tag{3.158}$$

Let us further write down the wave vectors of the incident, reflected and transmitted waves, assumed to be confined to the xz-plane, as

$$\mathbf{k}_i = k_1(\sin\theta_i \mathbf{a}_x + \cos\theta_i \mathbf{a}_z), \tag{3.159}$$

$$\mathbf{k}_r = k_1(\sin\theta_r \mathbf{a}_x - \cos\theta_r \mathbf{a}_z), \tag{3.160}$$

and

$$\mathbf{k}_t = k_2(\sin\theta_t \mathbf{a}_x + \cos\theta_t \mathbf{a}_z), \tag{3.161}$$

where

$$k_s = \omega\sqrt{\epsilon_s \mu_s} = \omega n_s/c; \qquad s = 1, 2.$$

Here

$$\boxed{n_s = \sqrt{\epsilon_s \mu_s/\epsilon_0 \mu_0}}, \tag{3.162}$$

is a **refractive index** in the sth medium. We can then infer from Eqs. (3.157) and (3.159) – (3.161) that the incidence and reflection angles must be the same,

$$\theta_i = \theta_r = \theta_1; \qquad \theta_t = \theta_2, \tag{3.163}$$

and the well-known Snell law for the incidence and transmission angles must hold

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}, \tag{3.164}$$

There are two possible polarizations: transverse magnetic (TM) or parallel and transverse electric (TE) or perpendicular which we are treating separately.

3.7.1 Transverse magnetic (TM) or parallel polarization

Consider first the TM case. The magnetic field is assumed to be polarized along y-axis, $\mathbf{H}_0 = H_0 \mathbf{a}_y$. The relevant Maxwell equations, corresponding to Fig. 3.17, are

$$\mathbf{k}_s \cdot \mathbf{H}_{0s} = 0; \qquad \mathbf{k}_s \cdot \mathbf{E}_{0s} = 0. \tag{3.165}$$

and

$$\mathbf{E}_{0s} = -\eta_s(\mathbf{a}_s \times \mathbf{H}_{0s}). \tag{3.166}$$

Here the indices 1 and 2, $(s = 1, 2)$, correspond to the media below and above the interface, respectively, and $\mathbf{a}_s = \mathbf{k}_s/k_s$. It follows from Maxwell's equations, Eqs. (3.165) and (3.166) that the electric and magnetic fields can be represented as

$$\mathbf{H}_i(\mathbf{r}, t) = H_{0i} \mathbf{a}_y \, e^{j(\mathbf{k}_i \cdot \mathbf{r} - \omega t)}$$
$$\mathbf{E}_i(\mathbf{r}, t) = \eta_1 H_{0i}(\mathbf{a}_x \cos\theta_1 - \mathbf{a}_z \sin\theta_1) \, e^{j(\mathbf{k}_i \cdot \mathbf{r} - \omega t)}, \tag{3.167}$$

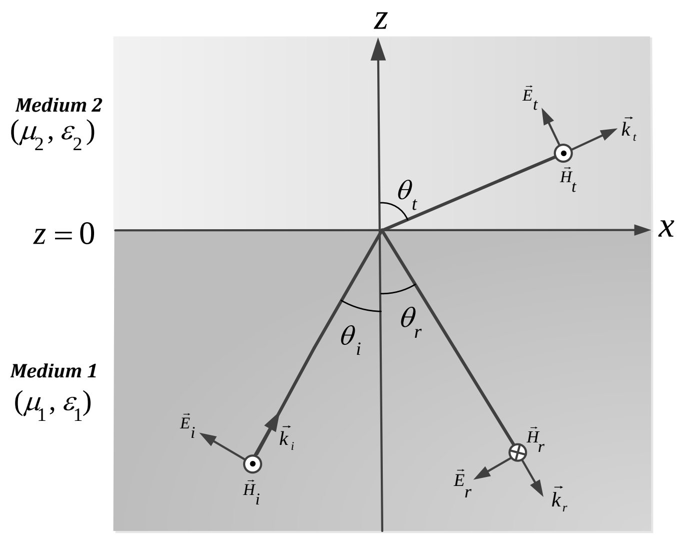

Figure 3.17: Oblique incidence of a TM plane wave.

for the incident wave;

$$\begin{aligned} \mathbf{H}_r(\mathbf{r},t) &= -H_{0r}\mathbf{a}_y\, e^{j(\mathbf{k}_r\cdot\mathbf{r}-\omega t)} \\ \mathbf{E}_r(\mathbf{r},t) &= \eta_1 H_{0r}(\mathbf{a}_x\cos\theta_1 + \mathbf{a}_z\sin\theta_1)\, e^{j(\mathbf{k}_r\cdot\mathbf{r}-\omega t)}, \end{aligned} \tag{3.168}$$

for the reflected wave and

$$\begin{aligned} \mathbf{H}_t(\mathbf{r},t) &= H_{0t}\mathbf{a}_y\, e^{j(\mathbf{k}_t\cdot\mathbf{r}-\omega t)} \\ \mathbf{E}_t(\mathbf{r},t) &= \eta_2 H_{0t}(\mathbf{a}_x\cos\theta_2 - \mathbf{a}_z\sin\theta_2)\, e^{j(\mathbf{k}_t\cdot\mathbf{r}-\omega t)}, \end{aligned} \tag{3.169}$$

for the transmitted wave. Here we assumed that a linearly polarized plane wave does not change its polarization upon reflection. As a result to keep the correct mutual orientation of $\mathbf{E}$, $\mathbf{k}$ and $\mathbf{H}$ we must assume that $\mathbf{H}$ changes its direction upon reflection, hence the minus sign in front of H_r in Eq. (3.168).

The boundary conditions for the tangential components of the fields across the interface state

$$H_{0i} - H_{0r} = H_{0t} \tag{3.170}$$

and

$$\eta_1 H_{0i}\cos\theta_1 + \eta_1 H_{0r}\cos\theta_1 = \eta_2 H_{0t}\cos\theta_2. \tag{3.171}$$

It then follows from Eqs. (3.170) and (3.171) that

$$H_{0r} = \frac{\eta_2\cos\theta_2 - \eta_1\cos\theta_1}{\eta_1\cos\theta_1 + \eta_2\cos\theta_2}\, H_{0i}, \tag{3.172}$$

and

$$H_{0t} = \frac{2\eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} H_{0i}. \tag{3.173}$$

Using (3.166) we arrive at the expressions for the electric fields in the form

$$E_{0i} = \eta_1 H_{0i}, \quad E_{0r} = \eta_1 H_{0r}, \quad E_{0t} = \eta_2 H_{0t}. \tag{3.174}$$

Finally, the complex reflection and transmission coefficients can be represented as

$$\boxed{r_{TM} = r_{\parallel} = \frac{E_{0r}}{E_{0i}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}}, \tag{3.175}$$

and

$$\boxed{t_{TM} = t_{\parallel} = \frac{E_{0t}}{E_{0i}} = \frac{2\eta_2 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}}. \tag{3.176}$$

Sometimes, the real transmission and reflection coefficients for energy fluxes are introduced as well by expressions

$$R_{TM} \equiv |r_{TM}|^2, \qquad T_{TM} \equiv |t_{TM}|^2. \tag{3.177}$$

Equations (3.175) and (3.176) are celebrated Fresnel formulas for the TM case.

As a limiting case, let us consider the case of normal incidence, $\theta_1 = \theta_2 = 0$. It then follows from Eqs. (3.175) – (3.177) that

$$r_n = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}, \qquad t_n = \frac{2\eta_2}{\eta_1 + \eta_2}. \tag{3.178}$$

Another instructive particular case corresponds to the Brewster angle θ_B at which $r_{TM} = 0$, i. e., there is no reflected TM wave. In this case,

$$r_{TM} = 0 \Longrightarrow \eta_2 \cos\theta_2 = \eta_1 \cos\theta_B,$$

and using Snell's law, we arrive at

$$\eta_2^2[1 - \sin^2\theta_B(\mu_1\epsilon_1/\mu_2\epsilon_2)] = \eta_1^2(1 - \sin^2\theta_B).$$

A simple algebra then leads to

$$\boxed{\sin\theta_B = \sqrt{\frac{1 - \mu_2\epsilon_1/\mu_1\epsilon_2}{1 - (\epsilon_1/\epsilon_2)^2}}}. \tag{3.179}$$

In a practically important case of nonmagnetic media, $\mu_1 = \mu_2 = \mu_0$,

$$\boxed{\tan\theta_B = n_2/n_1}. \tag{3.180}$$

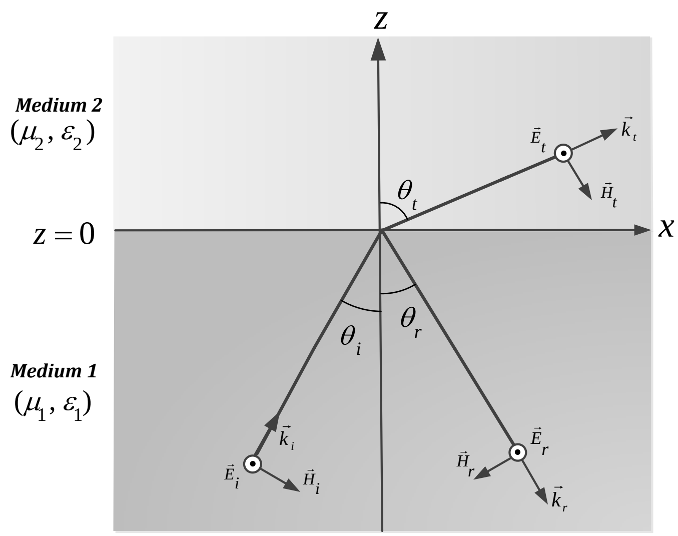

Figure 3.18: Oblique incidence of a TE plane wave.

3.7.2 Transverse electric (TE) or perpendicular polarization

In the TE case, $\mathbf{E}_0 = E_0\mathbf{a}_y$ and the magnetic field is given by

$$\mathbf{H}_{0s} = \frac{(\mathbf{a}_s \times \mathbf{E}_{0s})}{\eta_s}; \qquad s = 1, 2. \tag{3.181}$$

We thus obtain for the incident, reflected and transmitted fields the expressions

$$\mathbf{E}_\mathrm{i}(\mathbf{r}, t) = E_{0i}\mathbf{a}_y\, e^{j(\mathbf{k}_i\cdot\mathbf{r}-\omega t)}$$
$$\mathbf{H}_i(\mathbf{r}, t) = \frac{E_{0i}}{\eta_1}(-\mathbf{a}_x\cos\theta_1 + \mathbf{a}_z\sin\theta_1)\, e^{j(\mathbf{k}_i\cdot\mathbf{r}-\omega t)}, \tag{3.182}$$

$$\mathbf{E}_\mathrm{r}(\mathbf{r}, t) = E_{0r}\mathbf{e}_y\, e^{j(\mathbf{k}_r\cdot\mathbf{r}-\omega t)}$$
$$\mathbf{H}_r(\mathbf{r}, t) = \frac{E_{0r}}{\eta_1}(\mathbf{a}_x\cos\theta_1 + \mathbf{a}_z\sin\theta_1)\, e^{j(\mathbf{k}_r\cdot\mathbf{r}-\omega t)}, \tag{3.183}$$

and

$$\mathbf{E}_\mathrm{t}(\mathbf{r}, t) = E_{0t}\mathbf{a}_y\, e^{j(\mathbf{k}_t\cdot\mathbf{r}-\omega t)}$$
$$\mathbf{H}_t(\mathbf{r}, t) = \frac{E_{0t}}{\eta_2}(-\mathbf{a}_x\cos\theta_2 + \mathbf{a}_z\sin\theta_2)\, e^{j(\mathbf{k}_t\cdot\mathbf{r}-\omega t)}. \tag{3.184}$$

The continuity of tangential components of electric and magnetic fields across the interface leads to

$$E_{0i} + E_{0r} = E_{0t}, \tag{3.185}$$

and

$$\left(-\frac{E_{0i}}{\eta_1}+\frac{E_{0r}}{\eta_1}\right)\cos\theta_1=-\frac{E_{0t}}{\eta_2}\cos\theta_2. \tag{3.186}$$

Solving the last pair of equations, we arrive at the complex reflection and transmission coefficients in the form

$$\boxed{r_{\mathrm{TE}}=r_\perp=\frac{E_{0r}}{E_{0i}}=\frac{\eta_2\cos\theta_1-\eta_1\cos\theta_2}{\eta_1\cos\theta_2+\eta_2\cos\theta_1}}, \tag{3.187}$$

$$\boxed{t_{\mathrm{TE}}=t_\perp=\frac{E_{0t}}{E_{0i}}=\frac{2\eta_2\cos\theta_1}{\eta_1\cos\theta_2+\eta_2\cos\theta_1}}. \tag{3.188}$$

The Brewster angle is determined by the condition,

$$r_{\mathrm{TE}}=0, \tag{3.189}$$

implying that

$$\eta_2\cos\theta_{\mathrm{B}}=\eta_1\cos\theta_2, \tag{3.190}$$

or Using the Snell law, we can obtain the equation

$$\eta_2^2(1-\sin^2\theta_{\mathrm{B}})=\eta_1(1-\sin^2\theta_1). \tag{3.191}$$

Using the Snell law, we can obtain the expression

$$\sin\theta_{\mathrm{B}}=\sqrt{\frac{1-\mu_1\epsilon_2/\mu_2\epsilon_1}{1-(\mu_1/\mu_2)^2}}. \tag{3.192}$$

However, in virtually any practical case, the media under consideration are nonmagnetic, $\mu_1=\mu_2$. It then follows at once that Eq. (3.192) can never be satisfied resulting in the absence of the Brewster angle for the TE polarization.

Example 3. 13. A plane EM wave propagating in a lossless dielectric nonmagnetic medium with $\epsilon_r=9$, which occupies the half-space $z<0$, is impinged on a plane interface $z=0$ separating the medium from the other lossless nonmagnetic medium with $\epsilon_r=9/2$. The magnetic field of the incident wave is given, $\mathbf{H}(\mathbf{r},t)=\mathbf{a}_y\cos(10^9t-ax-a\sqrt{3}z)$. Determine (a) the angle of incidence, (b) the transmission angle, (c) the magnitude of a, (d) the wave polarization (TE/TM), and (e) the incident and reflected fields $\mathbf{E}_i(\mathbf{r},t)$ and $\mathbf{E}_r(\mathbf{r},t)$.

Solution. (a) First, $\mathbf{k}_i=a\mathbf{a}_x+a\sqrt{3}\mathbf{a}_z$*. The wave propagates at the angle* θ_1 *to the* z *axis such that* $\tan\theta_1=1/\sqrt{3}$*. Hence,* $sin\theta_1=1/2$*, implying that* $\underline{\theta_i=\theta_1=\pi/6}$ *from the Formula Sheet table. (b) Using Snell's law,* $\sin\theta_2=\sqrt{2}/2$*, implying that* $\underline{\theta_2=\pi/4}$*. (c) Next,* $k_1=\omega\sqrt{\epsilon_1\mu_1}=\omega\sqrt{\epsilon_{r1}}/c$*. Thus* $2a=10\Rightarrow\underline{a=5}$*. (d)* $\mathbf{H}$ *is normal to the plane of incidence, implying that the wave is TM polarized. (e)* $\mathbf{H}_i=Re[\mathbf{H}_{0i}e^{j(\omega t-\mathbf{k}\cdot\mathbf{r})}]$*, where* $\mathbf{H}_{0i}=\mathbf{a}_y$*. Thus,* $\mathbf{E}_{0i}=-\eta_1[\mathbf{a}_{ki}\times\mathbf{H}_0]$*, where* $\mathbf{a}_{ki}=\frac{1}{2}(\mathbf{a}_x+\sqrt{3}\mathbf{a}_z)$*. It follows that*

$$\mathbf{E}_{0i}=-\frac{\eta_0}{2\sqrt{\epsilon_{r1}}}[(\mathbf{a}_x+\sqrt{3}\mathbf{a}_z)\times\mathbf{a}_y]=20\pi(\sqrt{3}\mathbf{a}_x-\mathbf{a}_z),$$

which is purely real. Therefore,

$$\underline{\mathbf{E}_{\mathrm{i}}(\mathbf{r},t) = 20\pi(\sqrt{3}\mathbf{a}_x - \mathbf{a}_z)\cos(10^9 t - 5x - 5\sqrt{3}z)}.$$

Using the expression for the TM reflection coefficient with $\eta_1 = \eta_0/3$, $\eta_2 = \eta_0\sqrt{2}/3$, $\cos\theta_1 = \sqrt{3}/2$ *and* $\cos\theta_2 = \sqrt{2}/2$, *we obtain*

$$\underline{r_{\mathrm{TM}} = \frac{2-\sqrt{3}}{2+\sqrt{3}}}.$$

The reflected field is then

$$\underline{\mathbf{E}_{\mathrm{r}}(\mathbf{r},t) = 20\pi\frac{2-\sqrt{3}}{2+\sqrt{3}}(\sqrt{3}\mathbf{a}_x - \mathbf{a}_z)\cos(10^9 t - 5x + 5\sqrt{3}z)}.$$

3.7.3 Total internal reflection

Consider now the case of total internal reflection, $n_1 > n_2$. It follows from the Snell law that $\sin\theta_2 = (n_1/n_2)\sin\theta_1$, implying that for any angle greater than a certain critical angle,

$$\boxed{\theta_{\mathrm{c}} = \sin^{-1}(n_2/n_1)}, \tag{3.193}$$

the sine of the angle is greater than unity. The latter implies that θ_2 becomes complex with a purely imaginary cosine such that

$$\cos\theta_2 = j\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}. \tag{3.194}$$

We assume, for simplicity that the medium is nonmagnetic. It then readily follows from Eqs. (3.175) and (3.194) that

$$\overline{r}_{\mathrm{TM}} = \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}. \tag{3.195}$$

It can then be inferred from Eqs. (3.194) and (3.195) that the reflection amplitude of a totally reflected TM wave can be expressed as

$$\boxed{\overline{r}_{\mathrm{TM}} = e^{-2j\phi_{\mathrm{TM}}}}, \tag{3.196}$$

where the phase is given by

$$\boxed{\phi_{\mathrm{TM}} = \tan^{-1}\left(\frac{n_1\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_2^2\cos\theta_1}\right)}. \tag{3.197}$$

Using Eq. (3.194) in Eqs. (3.173) and (3.174), we can derive the expressions for complex amplitudes of the transmitted electric and magnetic fields as

$$\mathbf{E}_{0t}(z) = \frac{2E_{0i}(n_1^2/n_2)\cos\theta_1}{n_2\cos\theta_1 + jn_1\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}}\left[-\mathbf{a}_z\sin\theta_1 + j\mathbf{a}_x\left(\frac{n_2}{n_1}\right)\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}\right] \times\exp\left(-k_2 z\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}\right), \tag{3.198}$$

and

$$\mathbf{H}_{0t}(z) = \mathbf{a}_y\frac{2n_2H_{0i}\cos\theta_1}{n_2\cos\theta_1 + jn_1\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}} \times\exp\left(-k_2 z\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}\right). \tag{3.199}$$

Next, let us determine the magnitude and direction of the energy flow specified by the time-averaged Poynting vector,

$$\langle\boldsymbol{\mathcal{P}}_t(z)\rangle = \tfrac{1}{2}\mathrm{Re}[\mathbf{E}_{0t}(z)\times\mathbf{H}_{0t}^*(z)]. \tag{3.200}$$

We obtain, after some algebra, the following result

$$\langle\boldsymbol{\mathcal{P}}_t(z)\rangle_{\mathrm{TM}} = \frac{\mathbf{a}_x n_1^2|E_{0i}|^2\sin 2\theta_1\cos\theta_1}{\eta_1\left[n_2^2\cos^2\theta_1 + n_1^2\left(\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1\right)\right]}\exp\left(-2k_2 z\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}\right). \tag{3.201}$$

It can be concluded from Eq. (3.201) that the energy of the wave incident at an angle greater than the total internal reflection angle does not flow into the less optically dense medium. Rather, it propagates along the interface separating the two media, exponentially decaying in the direction normal to the interface.

By analogy with the TM case derivation, the reflection amplitude for total internal reflection of the TE polarization is given by

$$\boxed{\bar{r}_{\mathrm{TE}} = e^{-2j\phi_{\mathrm{TE}}}}, \tag{3.202}$$

where

$$\boxed{\phi_{\mathrm{TE}} = \tan^{-1}\left(\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}\right)}. \tag{3.203}$$

The corresponding expression for the Poynting vector is

$$\langle\boldsymbol{\mathcal{P}}_t(z)\rangle_{\mathrm{TE}} = \frac{\mathbf{a}_x n_1^3|E_{0i}|^2\sin 2\theta_1\cos\theta_1}{\eta_0(n_1^2 - n_2^2)}\exp\left(-2k_2 z\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}\right). \tag{3.204}$$

Chapter 4

Quasi-static fields

4.1 Quasi-static approximation

We consider time harmonic electromagnetic fields

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_\omega(\mathbf{r}) e^{-j\omega t}, \tag{4.1}$$

and

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_\omega(\mathbf{r}) e^{-j\omega t}. \tag{4.2}$$

Assuming the he medium is linear and isotropic, and nonmagnetic, but not necessarily homogeneous, we will have the constitutive relations for the harmonic components of the electric and magnetic flux and current densities as

$$\mathbf{D}_\omega(\mathbf{r}, \omega) = \epsilon(\mathbf{r}, \omega)\mathbf{E}_\omega(\mathbf{r}) \qquad \mathbf{B}_\omega(\mathbf{r}) = \mu_0 \mathbf{H}_\omega(\mathbf{r}), \tag{4.3}$$

and

$$\mathbf{J}_\omega(\mathbf{r}, \omega) = \sigma(\mathbf{r}, \omega)\mathbf{E}_\omega(\mathbf{r}). \tag{4.4}$$

With the aid of Eqs. (4.1) through (4.3), we obtain the Maxwell equations

$$\boxed{\nabla \cdot [\epsilon(\mathbf{r}, \omega)\mathbf{E}_\omega] = 0}, \tag{4.5}$$

$$\boxed{\nabla \cdot \mathbf{H}_\omega = 0}, \tag{4.6}$$

$$\boxed{\nabla \times \mathbf{E}_\omega = j\omega\mu_0 \mathbf{H}_\omega}, \tag{4.7}$$

and

$$\boxed{\nabla \times \mathbf{H}_\omega = \sigma(\mathbf{r}, \omega)\mathbf{E}_\omega - j\omega\epsilon(\mathbf{r}, \omega)\mathbf{E}_\omega}. \tag{4.8}$$

The quasi-static situation arises in two guises: the "traditional" (circuit theory) and "modern" (near-field). In particular, if one introduces a characteristic system size L, the quasi-static approximation is relevant whenever the phase an electromagnetic wave picks up on traversing the system is small compared to 2π, that is $kL \ll 2\pi$, which is synonymous with the following

$$\boxed{\frac{\nu L}{c} \ll 1}. \tag{4.9}$$

The criterion (4.9) works at sufficiently low frequencies, below 10 MHz, say, for typical circuit architectures of a few-centimeter to few-tens-of-centimeter sizes. Indeed, take, for example, $L \simeq 30$ cm and $\nu \simeq 10$ MHz, $\nu L/c \simeq \times 10^7 \times 30/3 \times 10^{10} \simeq 10^{-2} \ll 1$. This situation takes place in the low-frequency circuit theory. Another possibility arises whenever a system size is actually very small. For instance, for nanoscale systems, $L \simeq 10^{-9}$ m, (1 nm) even at optical frequencies $\nu \simeq 3 \times 10^{14}$ Hz, $\lambda = c/\nu \sim 10^{-6}$ m (1μm) the criterion (4.9) holds and it can be rewritten as

$$\boxed{\frac{L}{\lambda} \ll 1}. \tag{4.10}$$

Regardless of the actual physical situation, the quasi-static limit can be formally obtained by considering the limit of $\omega = 0$. In this case, Eqs. (4.5) to (4.8) reduce to

$$\boxed{\nabla \cdot [\epsilon_s(\mathbf{r})\mathbf{E}_s] = 0}, \tag{4.11}$$

$$\boxed{\nabla \cdot \mathbf{H}_s = 0}, \tag{4.12}$$

$$\boxed{\nabla \times \mathbf{E}_s = 0}, \tag{4.13}$$

$$\boxed{\nabla \times \mathbf{H}_s = \sigma_s(\mathbf{r})\mathbf{E}_s}. \tag{4.14}$$

and the actual fields $\mathbf{E}$ and $\mathbf{H}$ are given in terms of the static fields $\mathbf{E}_s$ and $\mathbf{H}_s$ as

$$\boxed{\mathbf{E} = \mathbf{E}_s e^{-j\omega t}, \qquad \mathbf{H} = \mathbf{H}_s e^{-j\omega t}}, \tag{4.15}$$

and the static conductivity $\sigma_s(\mathbf{r}) = \sigma(\mathbf{r}, 0)$ is introduced. The permittivity must also be replaced with its dc limiting value, $\epsilon_s(\mathbf{r}) = \epsilon(\mathbf{r}, 0)$.
Implication. In a quasi-static limit, spatial distributions of time-harmonic electric and magnetic fields are those dictated by electrostatics.

Since the static limit plays such an important role for low-frequency (or nanoscale) time-harmonic electromagnetic fields, we examine it more closely. Hereafter, we will drop the subscript "s". It follows at once from Eq. (4.13) and the fact that curl of a gradient is equal to zero that the electrostatic field can be expressed in terms of a gradient of a scalar function which will refer to as the **scalar potential** V as

$$\boxed{\mathbf{E} = -\nabla V}, \tag{4.16}$$

where the gradient can be expressed in the Cartesian coordinates as

$$\boxed{\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}}. \tag{4.17}$$

The minus sign in Eq. (4.16) is a matter of convention as we will see shortly. To provide a geometrical interpretation of the gradient, we examine the scalar field change as one moves from one point in space to another infinitesimally closed point. The situation is illustrated in Fig. 4.1. It follows that

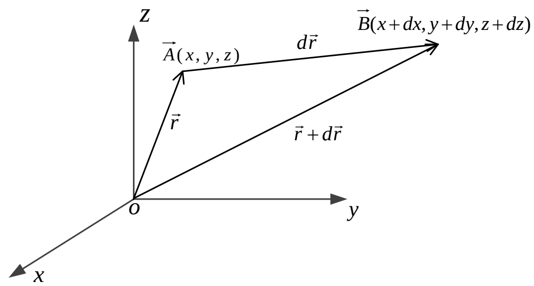

Figure 4.1: Illustrating the evaluation of the gradient.

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz. \tag{4.18}$$

On the other hand the infinitesimal distance between the points is

$$d\mathbf{r} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz. \tag{4.19}$$

It can be inferred from Eqs. (4.17) through (4.19) that

$$dV = \nabla V \cdot d\mathbf{r} = |\nabla V||d\mathbf{r}|\cos\phi, \tag{4.20}$$

where ϕ is the angle between the gradient vector and the displacement vector $d\mathbf{r}$. Consider now an equipotential surface on which $V = const$. For any two points on the equipotential surface, $dV = 0$. It then follows from Eq. (4.20) that the only way for the r.h.s to be equal to zero for any $d\mathbf{r}$, is $\cos\phi = 0$ implying that ∇V must be orthogonal to any displacement vector lying on the surface. Hence the gradient must be normal to the surface.

Geometrical interpretation of the gradient I. The gradient of a scalar potential is always normal to the surface.

As a result gradient can be used to determine a unit normal to a surface, see Fig 4.2. In the electromagnetic context, the electrostatic fields are always normal to the corresponding equipotential surfaces as is illustrated in Fig. 4.3.

Example. 4. 1. Find a unit normal to the surface $y = x^2$ at the point $(2, 4, 1)$.

Solution. The equation of the surface is $y - x^2 = 0$. One can then introduce a scalar field $\Phi(x, y) = y - x^2$ which is constant on the surface. Working out the gradient of Φ at $(2, 4, 1)$ gives, $\nabla\Phi = -2x\mathbf{a}_x + \mathbf{a}_y = -4\mathbf{a}_x + \mathbf{a}_y$. The unit normal is then $\mathbf{a}_n = \pm\nabla\Phi/|\nabla\Phi| = \pm(-4\mathbf{a}_x/\sqrt{17} + \mathbf{a}_y/\sqrt{17})$.

Next, dV is maximal in the direction in which ∇V is parallel to $d\mathbf{r}$. It can then be inferred that

Geometrical interpretation of the gradient II. Gradient of a scalar field points to

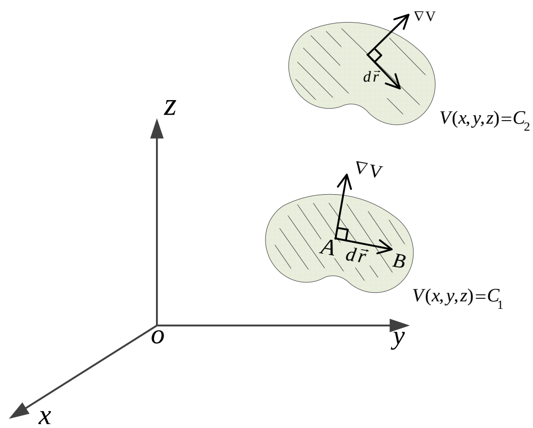

Figure 4.2: Illustrating the geometrical interpretation of the gradient.

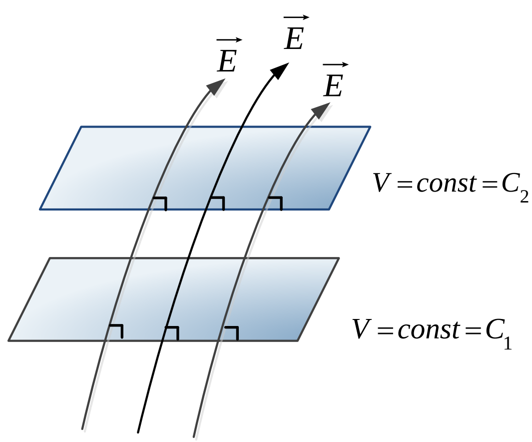

Figure 4.3: Electrostatic field and its equipotential surfaces.

the direction of the maximal change of the field.

Let us now briefly discuss the physical interpretation of the introduced scalar potential V. A point charge q, placed in an electrostatic field, experiences the force $\mathbf{F}_e = q\mathbf{E}$. The work done by an external agent to carry the charge from point A to point B in the field is

$$W_{AB} = -q\int_A^B d\mathbf{r}\cdot\mathbf{E} = q\int_A^B d\mathbf{r}\cdot\nabla V = q\int_A^B dV = V_B - V_A. \tag{4.21}$$

Thus, the work done by an external agent to move a unit charge between the points A and B is given by the potential difference:

$$W_{AB}/q = V_B - V_A. \tag{4.22}$$

Clearly, if $V_B > V_A$, $W_{AB} > 0$, the external agent must do (positive) work to bring the charge to a point with a higher potential (just like we have to work–and sweat–to elevate an object in the gravitational field of earth). We chose a minus sign in Eq. (4.23) to conform to this (physically meaningful) convention. Although it is the potential difference that has a direct physical interpretation, one can also introduce a potential at a given point relative to a reference point. It is convenient to take the latter be far removed, effectively be in infinity and let $V_\infty = 0$. It then follows that the work done by an external agent to bring a unit charge from infinity determines the potential,

$$\boxed{V = W_\infty/q}. \tag{4.23}$$

There are two important corollaries of Eq. (4.16) and (4.21)

Corollary 1. The work done to move a charge in an electrostatic field does not depend on the path.

Corollary 2. The work done to move a charge in an electrostatic field along a closed path is zero.

The first corollary follows at once from the fact that the performed work depends only on the potential difference between the two points, not on the actual path. The second flows from the first by noticing that if the path is closed, the end points must be the same resulting in zero overall work. Both corollaries are useful for solving practical problems.

Example 4. 2. Determine the work done to move a unit charge in the field $\mathbf{E} = \mathbf{a}_x y + \mathbf{a}_y x$ from point $(1, 2, 1)$ to the point $(3, 4, 1)$.

Solution. The field is clearly electrostatic because $\nabla \times \mathbf{E} = 0$. *Since the work does not depend on the path, we break the path into two intervals,* $(1, 2, 1) \to (3, 2, 1)$ *and* $(3, 2, 1) \to (3, 4, 1)$. *On the first interval,* $d\mathbf{l} = \mathbf{a}_x dx$ *and* $W_1 = -\int_1^3 y dx = -2\times 2 = -4$ *J. On the second interval,* $d\mathbf{l} = \mathbf{a}_y dy$ *and* $W_2 = -\int_2^4 dyx = 3 \times 2 = -6$ *J. The total work is then* $W = W_1 + W_2 = -10$ *J.*

Alternatively, we can first determine the potential,

$$E_x = y = -\frac{\partial V}{\partial x}, \qquad E_y = x = -\frac{\partial V}{\partial y}. \tag{4.24}$$

It follows by integrating the first equation that $V(x, y) = -xy + F(y)$ *where* F *is to be determined. Substituting this into the second equation, we arrive at* $F'(y) = 0 \Rightarrow F(y) = const$. *Thus* $V(x, y) = -xy + const$. *The work done to move a unit charge can then be worked out from the definition:* $W = V_2 - V_1 = -12 + 2 = -10.$ *J.*

Finally, we observe that by combining Eqs. (4.16) and (4.11), we obtain the general quasi-static Laplace's equation

$$\boxed{\nabla \cdot [\epsilon_s(\mathbf{r})\nabla V] = 0}. \tag{4.25}$$

For homogeneous media, $\epsilon_s = const$, the general Laplace's equation simplifies to

$$\boxed{\nabla^2 V = 0}, \tag{4.26}$$

where the Laplacian on the l.h.s. is defined in the Cartesian coordinates as

$$\boxed{\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}. \tag{4.27}$$

Eqs (4.25) or (4.26) allow one to reduce the problem of finding an unknown vector field to that of figuring out an unknown scalar field which has one third as many unknown variables!

4.2 Capacitance, conductance and inductance

We now introduce several lumped circuit parameters that are frequently encountered in low-frequency situations.

Definition. An arrangement of two identical conductors carrying equal and opposite charges which are separated by a dielectric medium is called a **capacitor**, see Fig. 4.4. The conductors are referred to as **capacitor plates**. The **capacitance** C of a capacitor

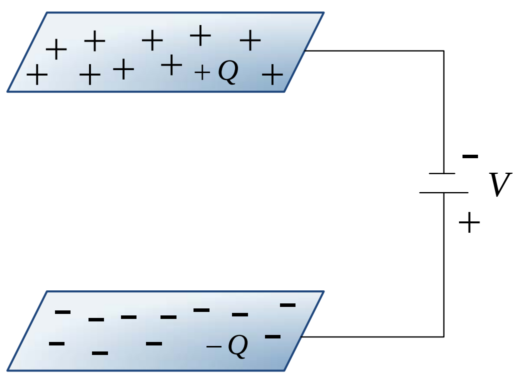

Figure 4.4: Illustrating a generic capacitor.

carrying a charge Q for given voltage V_0 between the plates is defined as the ratio

$$\boxed{C = \frac{Q}{V_0}}. \tag{4.28}$$

Capacitors are used to store charges. In this connection a natural question arises: What is the electrostatic energy stored in a capacitor? To answer the question, we shall determine the energy of a charged system. By the energy conservation law, the latter is equal to the work done to assemble a given charge configuration.

We consider first a simple system containing just three point charges, Q_1, Q_2, and Q_3, located at the points P_1, P_2, and P_3 with the potentials V_1, V_2, and V_3, respectively. There is no work done to bring charge Q_1 from infinity to the point P_1 since there are no other charges to influence such a move. Thus, $W_1 = 0$. However, the work done to move charge Q_2 from infinity to its location is $W_2 = Q_2 V_{12}$ where V_{12} is the potential generated at the position P_2 by the charge at the position P_1. In general, we can denote the potential generated by the i'th charge (i=1,2,3) at the position P_j, (j=1,2,3) by V_{ij}.

By the same token, the work needed to be done to move charge Q_3 in the position will be $W_3 = Q_3(V_{13} + V_{23})$. The overall work can then be expressed as

$$W_e = 0 + Q_2 V_{12} + Q_3(V_{13} + V_{23}). \tag{4.29}$$

If we reverse the order, we can represent the same amount of work as

$$W_e = 0 + Q_2 V_{32} + Q_1(V_{21} + V_{31}) \tag{4.30}$$

It follows by adding Eqs. (4.29) and (4.30) term by term that

$$2W_e = Q_1 V_1 + Q_2 V_2 + Q_3 V_3, \tag{4.31}$$

where V_j is the total potential at the point P_j. It follows that

$$W_e = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3). \tag{4.32}$$

Generalizing to N charges we obtain

$$\boxed{W_e = \tfrac{1}{2}\sum_{k=1}^{N} Q_k V_k}, \tag{4.33}$$

which is the interaction energy of a system of point charges. For a continuous charge distribution, the interaction energy can be further generalized to yield

$$\boxed{W_e = \tfrac{1}{2}\int_v dv \rho_v V}, \tag{4.34}$$

for a volume charge and

$$\boxed{W_e = \tfrac{1}{2}\int_S dS \rho_S V}, \tag{4.35}$$

for a surface charge distribution.

We can now express the energy stored in a capacitor in terms of its capacitance. We begin by writing down the interaction energy of the charges on both plates as

$$W_e = \tfrac{1}{2}(\int_{S_1} dS\, \rho_{S1} V_1 + \int_{S_2} dS\, \rho_{S2} V_2). \tag{4.36}$$

In Eq. (4.36), ρ_{S1} and ρ_{S2} are surface charge densities on each plate. Since in the static situation, there are no currents on the plates, each plate is an equipotential surface and we can factor out V_1 and V_2 from the integrands on the r.h.s of Eq. (4.36), leading to

$$W_e = \tfrac{1}{2}(V_1 \int_{S_1} dS\, \rho_{S1} + V_2 \int_{S_2} dS\, \rho_{S2}). \tag{4.37}$$

Next, the total charges on the plates are equal and opposite, implying that

$$\int_{S_1} dS\, \rho_{S1} = -\int_{S_2} dS\, \rho_{S2} = Q. \tag{4.38}$$

It can be inferred from Eqs. (4.37) and (4.38) that

$$\boxed{W_e = \tfrac{1}{2}QV_0 = \tfrac{1}{2}CV_0^2 = \tfrac{1}{2}Q^2/C}, \qquad (4.39)$$

where $V_0 = V_1 - V_2$ is the voltage across the plates.

On the other hand, the assembled charges generate the electrostatic field inside the capacitor with the energy

$$W_e = \tfrac{1}{2}\int_v dv\epsilon E^2. \qquad (4.40)$$

By energy conservation,

$$\tfrac{1}{2}CV^2 = \tfrac{1}{2}\int_v dv\epsilon E^2, \Longrightarrow \boxed{C = \frac{1}{V_0^2}\int_v dv\epsilon E^2}. \qquad (4.41)$$

Eq. (4.41) provides a practical way to calculate the capacitance of any electrostatic or quasi-electrostatic system. The algorithm is rather simple:

1. Determine the scalar potential solving the Laplace equation (4.26);
2. Find the electric field with the help of Eq. (4.16);
3. Use Eq. (4.41) to figure out the capacitance.

The formula stemming from the energy balance consideration is actually much simpler to use than the definition of capacitance (4.28).

In cases involving currents, there are two more important circuit parameters.

Definition. Conductance G is the ration of the total current through the conductor to the voltage between the conductor ends,

$$\boxed{G = \frac{I}{V_0}}. \qquad (4.42)$$

In case all currents are conduction currents, Eq. (4.42) reduces to

$$\boxed{G = \frac{1}{V_0}\int_S (d\mathbf{S}\cdot\mathbf{E})\sigma}. \qquad (4.43)$$

To find the conductance for a given voltage between the conductor ends you can

1. Determine the scalar potential;
2. Find the electric field and the corresponding current density;
3. Find the total current through the conductor cross-section;
4. Apply Eq. (4.42) to obtain the conductance.

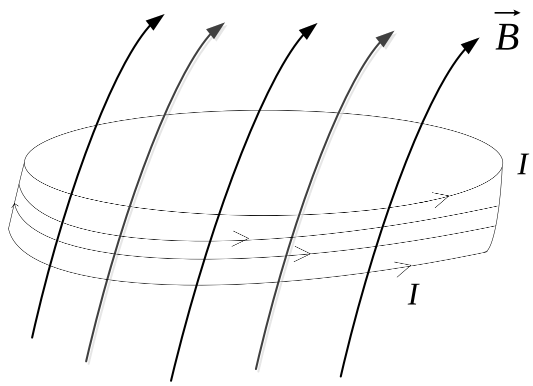

Figure 4.5: Illustrating a generic indictor.

Definition. **Inductance** L is the ratio of the magnetic linkage through a closed loop (or loops) to the current circulating in the loop(s). The conductors carrying currents, which generate magnetic fluxes through their cross-sections, are known as **inductors**, see Fig. 4.5.
The inductance is then defined as

$$\boxed{L = \frac{\Psi}{I} = \frac{N}{I}\int_S d\mathbf{S}\cdot\mathbf{B}}, \tag{4.44}$$

where N is the number loops placed in close proximity of each other. In case the current is due to conduction only,

$$\boxed{L = \frac{N\int_S d\mathbf{S}\cdot\mathbf{B}}{\int_S (d\mathbf{S}\cdot\mathbf{E})\sigma}}. \tag{4.45}$$

Similar to the capacitance calculation, it is useful to apply energy balance considerations to work out inductances of various inductors. One the one hand, we know from the circuit theory that the energy associated with each inductor is

$$W_m = \tfrac{1}{2}LI^2. \tag{4.46}$$

On the other hand, this energy is stored in the magnetic field of the inductor, hence

$$W_m = \tfrac{1}{2}\int_v dv\,\mu H^2. \tag{4.47}$$

Eqs. (4.46) and (4.47) imply that

$$\boxed{L = \frac{1}{I^2}\int_v dv\,\mu H^2}. \tag{4.48}$$

Thus to determine the inductance of an inductor one can

1. Find the scalar potential;
2. Determine the field and corresponding current density;
3. Calculate the total current through the conductor cross-section;
4. Solve Ampère's equation (4.14) to determine the magnetic field;
5. Apply Eq. (4.48) to find the inductance.

We will now illustrate the use of Eqs. (4.41), (4.43), and (4.48) to calculate some circuit parameters of a simple arrangement shown in Fig. 4.6.

Example 4. 3. Determine the capacitance and conductance of the system shown in Fig. 4.6 at low frequencies. The voltage between the plates is $V_0 \cos \omega t$. The permittivity and conductivity of the material between the capacitor plates are ϵ and σ, respectively.

Solution. We look for quasi-static solutions for the fields with the harmonic time dependence. The spatial field distributions can be found by solving the static Maxwell's equations. It follows from the problem symmetry that the potential must depend only on the x-coordinate. Thus,

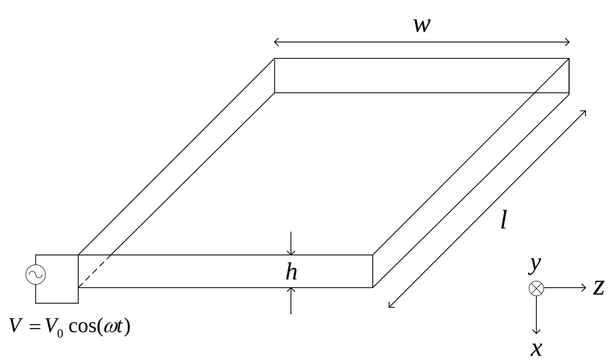

Figure 4.6: Illustrating the arrangement for Example 4.3.

$$\frac{d^2 V}{dx^2} = 0, \tag{4.49}$$

subject to the boundary conditions, $V(x = h) = 0$ and $V(x = 0) = V_0$. The solution to Eq. (4.49) subject to the boundary conditions is

$$V(x) = V_0(h - x)/h, \tag{4.50}$$

implying that

$$\mathbf{E} = -\nabla V = \mathbf{a}_x V_0/h. \tag{4.51}$$

It then follows that the electrostatic energy between the capacitor plates is

$$W_e = \int_v dv \epsilon E^2/2 = \frac{\epsilon V_0^2}{2h^2} wlh. \tag{4.52}$$

On the other hand,

$$W_e = \tfrac{1}{2}CV_0^2, \tag{4.53}$$

implying that

$$\underline{C = \frac{\epsilon w l}{h}}. \tag{4.54}$$

Next, the current density,

$$\mathbf{J} = \sigma\mathbf{E} = \mathbf{a}_x \sigma V_0/h. \tag{4.55}$$

The total current between the plates is then,

$$I = \int_S d\mathbf{S}\cdot\mathbf{J} = \sigma V_0 w l/h. \tag{4.56}$$

Thus,

$$\underline{G = \frac{\sigma w l}{h}}. \tag{4.57}$$

Chapter 5

Applications

5.1 Transmission Lines

5.1.1 Transmission line equations

Transmission lines (TL) are used to guide electromagnetic wave propagation to enhance efficiency of power delivery from a transmitter to a receiver(s). There are four major types of transmission lines, see Fig. 5.1

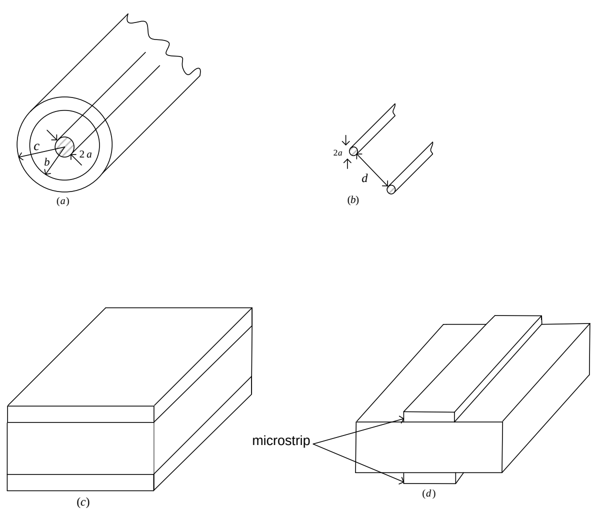

Figure 5.1: Types of TLs: (a) Coaxial cable, (b) two-wire line, (c) parallel-plate, and (d) microstrip line.

- Parallel-plate lines, also referred to as strip-lines;
- Two-wire lines (power lines or telephone lines);

- Coaxial lines (TV cables, electronic cables);
- Microstrip lines.

All can be characterized in terms of the voltage between the two conductors and the current along the conductors as is schematically displayed in Fig. 5.2. In general,

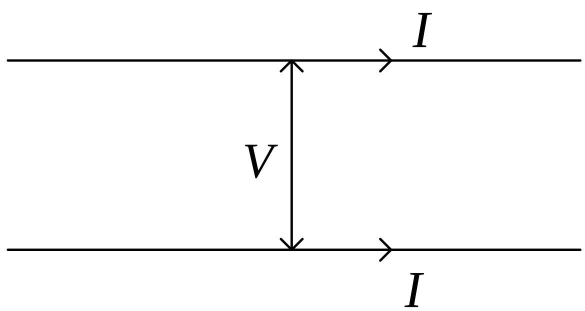

Figure 5.2: Transmission line description in terms of the voltage across and current along the line.

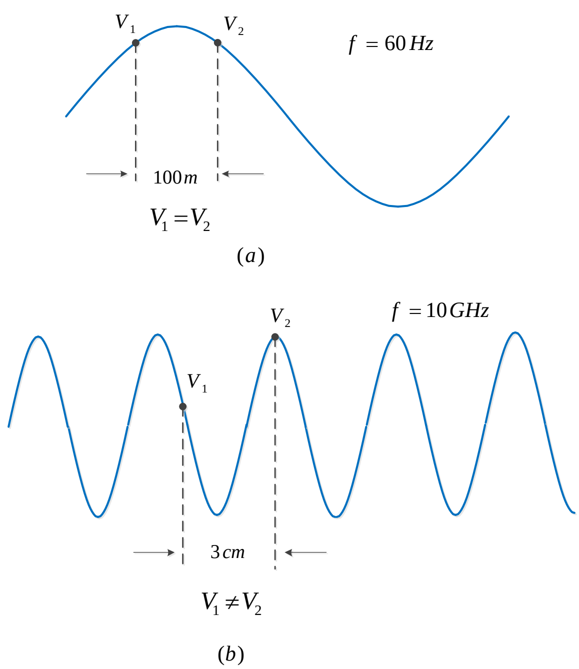

Figure 5.3: Illustrating low-and high-frequency regimes of TL operation.

the quasi-static approximation only applies to transmission lines operating at relatively low frequencies. As an example, take the operating frequency of a few tenths of hertz, $f = 60$ Hz, say. The operating wavelength, associated with this frequency, is $\lambda \simeq c/f = 5000$ km. Thus, for a section of the transmission line of $l = 100$ m, $l/\lambda = 2 \times 10^{-5} \ll 1$. The quasi-static approximation works well: $V_1 = V_2$ along the entire section. This situation is displayed in Fig. 5.3(a). On the other hand, for a transmission line operating at $f = 10$ GHz, say, $\lambda \simeq c/f = 3$ cm. For a three-centimeter long line section, $l/\lambda = 1$ and $V_1 \neq V_2$ for any two points within this section as is depicted in Fig. 5.3(b).

All transmission lines are characterized by four circuit parameters:

- R is a finite resistance per unit length Ω/m along the current carrying (imperfect) conductors;
- G is a finite conductance per unit length S/m between the two (imperfect) conductors making up the line;
- C is a finite capacitance of the line per unit length, F/m;
- L is the finite inductance of the line per unit length, H/m.

The transmission line equations can be derived from the circuit model displayed in Fig. 5.4.

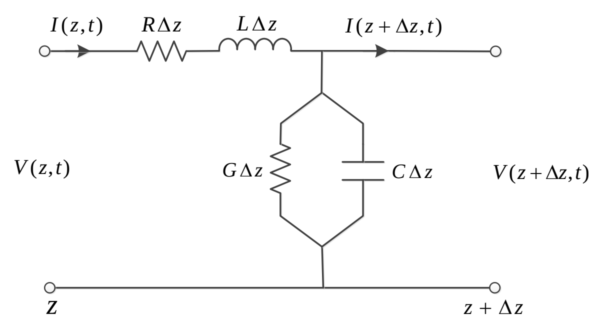

Figure 5.4: An equivalent circuit model for TL.

First, applying the voltage Kirchoff's law to the closed circuit, we obtain

$$v(z,t) = i(z,t)\, R\Delta z + L\,\Delta z \frac{\partial i(z,t)}{\partial t} + v(z+\Delta z, t). \tag{5.1}$$

It follows from Eq. (5.1) that

$$-\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = R\, i(z,t) + L\,\frac{\partial i(z,t)}{\partial t}. \tag{5.2}$$

In the limit $\Delta z \to 0$, we arrive at the first transmission line equation

$$\boxed{\frac{\partial v}{\partial z} = -R\, i - L\,\frac{\partial i}{\partial t}}. \tag{5.3}$$

Applying the current Kirchoff's law to the circuit node in Fig. 5.4, we obtain

$$i(z,t) = v(z,t)G\Delta z + C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} + i(z+\Delta z,t). \tag{5.4}$$

Eq. (5.4) can be rearranged as

$$-\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = G\, i(z,t) + C\, \frac{\partial v(z+\Delta z,t)}{\partial t}. \tag{5.5}$$

In the limit $\Delta z \to 0$, we arrive at the second transmission line equation

$$\boxed{\frac{\partial i}{\partial z} = -G\, v - C\, \frac{\partial v}{\partial t}}. \tag{5.6}$$

Eqs. (5.3) and (5.6) are the generic transmission line equations.

We now seek time-harmonic solutions to Eqs. (5.3) and (5.6) in the form

$$v(z,t) = V(z)e^{j\omega t}, \qquad i(z,t) = I(z)e^{j\omega t}. \tag{5.7}$$

On substituting from Eq. (5.7) into the transmission line equations, we obtain

$$\frac{dV}{dz} = -RI(z) - j\omega LI(z) = -ZI(z), \tag{5.8}$$

and

$$\frac{dI}{dz} = -GV(z) - j\omega CV(z) = -YV(z). \tag{5.9}$$

Here we introduced

- $\boxed{Z = R + j\omega L}$, the complex impedance per unit length, Ω/m,
- $\boxed{Y = G + j\omega C}$, the complex admittance per unit length, S/m.

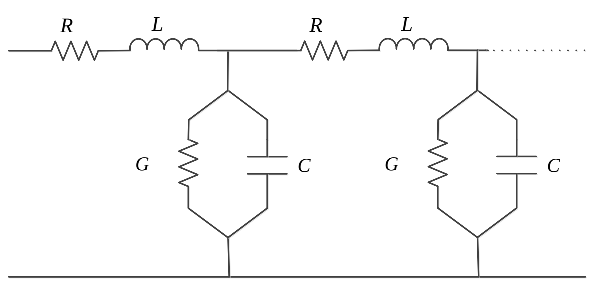

Figure 5.5: Circuit representation for a general TL.

The general solutions to Eqs. (5.8) and (5.9) can be obtained by eliminating one of the variables in favor of the other. For instance, eliminating the current, we arrive at

$$\frac{d^2V}{dz^2} - \gamma^2 V = 0, \qquad \gamma = \sqrt{YZ} = \alpha + j\beta, \tag{5.10}$$

with the solution,

$$V(z) = \underbrace{V_0^{(+)} e^{-\gamma z}}_{\text{incident wave}} + \underbrace{V_0^{(-)} e^{\gamma z}}_{\text{reflected wave}} . \tag{5.11}$$

Here γ is a complex propagation constant. The forward-propagating part corresponds to the incident wave, while the backward-propagating one to the reflected wave. It can be inferred from Eq. (5.8) that

$$I(z) = -\frac{1}{Z}\frac{dV}{dz}. \tag{5.12}$$

On the other hand, a general solution for the current is

$$I(z) = \underbrace{I_0^{(+)} e^{-\gamma z}}_{\text{incident wave}} + \underbrace{I_0^{(-)} e^{\gamma z}}_{\text{reflected wave}} . \tag{5.13}$$

Eqs. (5.11) through (5.13) imply that

$$I_0^{(+)} = V_0^{(+)}/Z_0, \qquad I_0^{(-)} = -V_0^{(-)}/Z_0, \tag{5.14}$$

where

$$\boxed{Z_0 = \sqrt{\frac{Z}{Y}}}, \tag{5.15}$$

is the characteristic impedance of the transmission line.

Consider now two important particular cases.

1. Lossless TL:
$$R = G = 0. \tag{5.16}$$
In this case, $\alpha = 0$ and $\beta = \omega\sqrt{LC}$. It follows that the phase velocity, $v_p = \omega/\beta = 1/\sqrt{LC}$. Also, the TL impedance, $Z_0 = \sqrt{L/C}$.

2. Distortionless TL:
$$\frac{R}{L} = \frac{G}{C}. \tag{5.17}$$
In this case,
$$\alpha = R\sqrt{C/L}, \qquad \beta = \omega\sqrt{LC}. \tag{5.18}$$
It is again seen that there is no dispersion, $v_p = \omega/\beta = 1/\sqrt{LC}$, and the TL impedance is the same as that for a lossless line, $Z_0 = \sqrt{L/C}$. However, a distortionless line need not have zero loss, and hence it is a more realistic case. In practice, telephone lines are required to be distortionless. Lossless lines–or the closest approximation available in reality–are desirable for power transmission.

Example 5. 1. Show that at high frequencies $R \ll \omega L$ and $G \ll \omega L$,

$$\gamma \simeq \left(\frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}\right) + j\omega\sqrt{LC}.$$

Obtain a similar formula for Z_0.
Solution. Use the definition, $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$. *It follows that*

$$\gamma = j\omega\sqrt{LC}\sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)}.$$

Expanding in powers of small parameters $j\omega L/R$ *and* $j\omega C/G$ *using*

$$(1 + \epsilon)^n \approx 1 + n\epsilon, \quad \textit{for} \quad \epsilon \ll 1,$$

we obtain

$$\gamma \simeq j\omega\sqrt{LC}\left(1 + \frac{1}{2}\frac{R}{j\omega L}\right)\left(1 + \frac{1}{2}\frac{G}{j\omega C}\right),$$

and keeping only first terms in both small parameters, we obtain

$$\gamma \simeq j\omega\sqrt{LC}\left(1 + \frac{1}{2}\frac{R}{j\omega L} + \frac{1}{2}\frac{G}{j\omega C}\right) \simeq \underline{j\omega\sqrt{LC} + \left(\frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}\right)}.$$

By the same token,

$$Z_0 = \sqrt{\frac{L}{C}}\sqrt{\frac{1 + R/j\omega L}{1 + G/j\omega C}} \simeq \sqrt{\frac{L}{C}}\sqrt{1 + \frac{R}{j\omega L} - \frac{G}{j\omega C}}.$$

Thus,

$$\underline{Z_0 \simeq \sqrt{\frac{L}{C}}\left(1 - j\frac{R}{2\omega L} + j\frac{G}{2\omega C}\right)}.$$

Example 5. 2. Given Z_0, α, and β of a lossy transmission line. Assuming that Z_0 is purely real, determine R, L, C and G.
Solution. $\gamma = \alpha + j\beta$. *Observe that*

$$\gamma = \sqrt{xy}, \qquad Z_0 = \sqrt{x/y},$$

where $x = R + j\omega L$ *and* $y = G + j\omega C$. *It follows that*

$$\gamma^2 = xy, \qquad Z_0^2 = x/y.$$

Hence,

$$y = \gamma/Z_0, \qquad x = \gamma Z_0.$$

Separating real and imaginary parts, we obtain the TL parameters,

$$\underline{R = Re(\gamma Z_0) = \alpha Z_0}, \qquad \underline{G = Re(\gamma/Z_0) = \alpha/Z_0}.$$

and

$$\underline{L = Im(\gamma Z_0/\omega) = \beta Z_0/\omega}, \qquad \underline{C = Im\left(\frac{\gamma}{\omega Z_0}\right) = \frac{\beta}{\omega Z_0}}.$$

5.1.2 Input impedance, reflection coefficient, and power

Recall that the time-harmonic waves on a transmission line are determined in terms of the voltages and currents:

$$V(z) = V_0^{(+)} e^{-\gamma z} + V_0^{(-)} e^{\gamma z}. \tag{5.19}$$

and

$$I(z) = \frac{V_0^{(+)}}{Z_0} e^{-\gamma z} - \frac{V_0^{(-)}}{Z_0} e^{\gamma z}. \tag{5.20}$$

If the line is loaded at $z = l$ to the load with the impedance Z_L as is shown in Fig. 5.6,

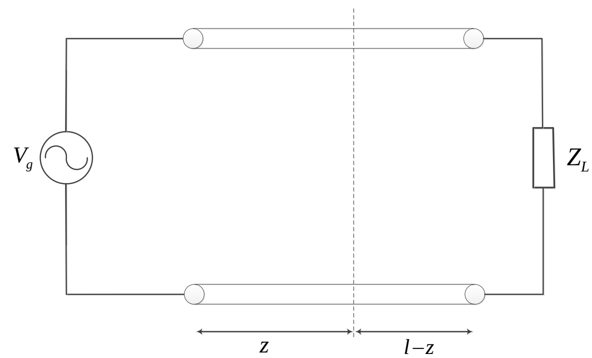

Figure 5.6: TL connected to a load with the complex impedance Z_L.

the boundary conditions are

$$V_L = V(z = l), \qquad I_L = I(z = l). \tag{5.21}$$

It can be inferred from Eqs. (5.19), (5.20) and (5.21) that

$$V_0^{(+)} = \tfrac{1}{2}(V_L + Z_0 I_L) e^{\gamma l}, \tag{5.22}$$

and

$$V_0^{(-)} = \tfrac{1}{2}(V_L - Z_0 I_L) e^{-\gamma l}. \tag{5.23}$$

Let us now determine the input impedance at a general position z along the TL. We define the input impedance as

$$Z_{\rm in}(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^{(+)} e^{-\gamma z} + V_0^{(-)} e^{\gamma z}}{V_0^{(+)} e^{-\gamma z} - V_0^{(-)} e^{\gamma z}}. \tag{5.24}$$

Using Eqs. (5.22) and (5.23) in Eq. (5.24), we arrive at

$$Z_{\rm in}(z) = Z_0 \frac{(V_L + Z_0 I_L) e^{\gamma(l-z)} + (V_L - Z_0 I_L) e^{-\gamma(l-z)}}{(V_L + Z_0 I_L) e^{\gamma(l-z)} - (V_L - Z_0 I_L) e^{-\gamma(l-z)}}. \tag{5.25}$$

Using the hyperbolic functions,

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}), \tag{5.26}$$

and

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{5.27}$$

we obtain the expression for the input impedance at the position z on the line as

$$\boxed{Z_{\text{in}}(z) = Z_0 \left[\frac{Z_L + Z_0 \tanh \gamma(l-z)}{Z_0 + Z_L \tanh \gamma(l-z)} \right]}. \tag{5.28}$$

Here $Z_L = V_L/I_L$ is the load impedance. For a lossless line, $\gamma = j\beta$ and $\tanh[j\beta(l - z)] = j\tan[\beta(l-z)]$, implying that

$$\boxed{Z_{\text{in}}(z) = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta(l-z)}{Z_0 + jZ_L \tan \beta(l-z)} \right]}. \tag{5.29}$$

Let us consider three particularly important limiting cases of lossless lines:

- Short-circuited line, $Z_L = 0, \Longrightarrow Z_{sc}(0) = jZ_0 \tan \beta l$;
- Open-circuited line, $Z_L = \infty, \Longrightarrow Z_{oc}(0) = \frac{Z_0}{j \tan \beta l} = -jZ_0 \cot \beta l$;
- Matched line, $Z_L = Z_0, \Longrightarrow Z_m(z) = Z_0$.

Example 5. 3. Show that a lossy transmission line of length l has an input impedance at the generator $Z_{sc} = Z_0 \tanh \gamma l$ when shorted and $Z_{oc} = Z_0 \coth \gamma l$ when open.
Solution. At the generator, $z = 0$ and $Z_{in}/Z_0 = (Z_L + Z_0 \tanh \gamma l)/(Z_0 + Z_L \tanh \gamma l)$. It follows that as $Z_L = 0$, $Z_{sc} = Z_0 \tanh \gamma l$ and as $Z_L = \infty$, $Z_{oc} = Z_0/\tanh \gamma l = Z_0 \coth \gamma l$.

Next, we introduce the reflection coefficient as a ratio of the reflected voltage to the incident voltage at the position z on the line,

$$\Gamma(z) = \frac{V_0^{(-)} e^{\gamma z}}{V_0^{(+)} e^{-\gamma z}}. \tag{5.30}$$

It can be inferred using Eqs. (5.22), (5.23) and (5.30) that

$$\boxed{\Gamma(z) = \frac{Z_L - Z_0}{Z_L + Z_0} e^{2\gamma(z-l)}}. \tag{5.31}$$

Finally, we will determine the average power transmitted by the line from the source to a receiver by a lossless transmission line. The average power at a distance l away from the generator is

$$\langle P \rangle = \tfrac{1}{2}\text{Re}[V(l) I^*(l)]. \tag{5.32}$$

Using Eqs. (5.19) and (5.20), and assuming a lossless line, $\gamma = j\beta$, we obtain

$$\begin{aligned} \langle P \rangle &= \tfrac{1}{2}\mathrm{Re}\left\{ V_0^{(+)} \left[e^{j\beta l} + \Gamma(l) e^{-j\beta l} \right] \tfrac{V_0^{(+)*}}{Z_0} \left[e^{-j\beta l} - \Gamma^*(l) e^{j\beta l} \right] \right\} \\ &= \tfrac{1}{2}\mathrm{Re}\left\{ \tfrac{|V_0^{(+)}|^2}{Z_0} \left[1 - |\Gamma(l)|^2 + \Gamma(l) e^{-2j\beta l} - \Gamma^*(l) e^{2j\beta l} \right] \right\}. \end{aligned} \quad (5.33)$$

Since the last two terms are purely imaginary, we arrive at our result for the transmitted power

$$\boxed{\langle P \rangle = \frac{|V_0^{(+)}|^2}{2Z_0} \left[1 - |\Gamma(l)|^2 \right]}. \quad (5.34)$$

5.2 Optical fibers

Optical fibers (OF) serve as the most favorable platform for modern communications systems.Their main advantages over the competition are as follows

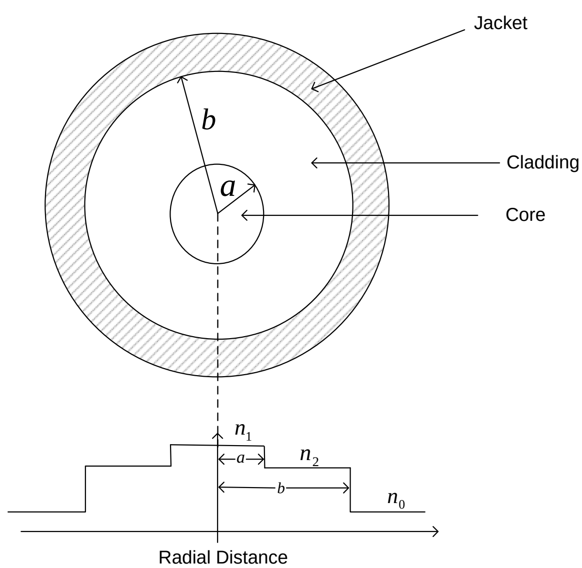

Figure 5.7: Schematic sketch of an optical fiber.

1. The OFs have a very large bandwidth, typically of a few THz at optical frequencies of about 100 THz;

2. The OFs have extremely low losses of a few tenths of dBs per kilometer;

3. The OFs are immune to many common noise sources that plague conventional communication systems;

4. The OFs are quite secure;
5. The OF manufacturing is a standard low-cost technology.

A typical OF is made of–typically silica glass–core surrounded by a cladding of a material with a slightly lower index of refraction. The whole system is then wrapped into a jacket as is shown in Fig. 5.7. Recall from Chap. 3 that for nonmagnetic media with the constitutive parameters, ϵ and $\mu = \mu_0$, the refractive index is defined as

$$n = \sqrt{\epsilon/\epsilon_0}. \tag{5.35}$$

Recall further that any plane wave striking the interface between the fiber and cladding at the angle greater than the total internal reflection angle,

$$\theta_c = \sin^{-1}(n_2/n_1), \tag{5.36}$$

is totally reflected and thereby is trapped by the fiber core. Hence light is transmitted along the fiber because of total internal reflection from the core-cladding interface, see Fig. 5.8.

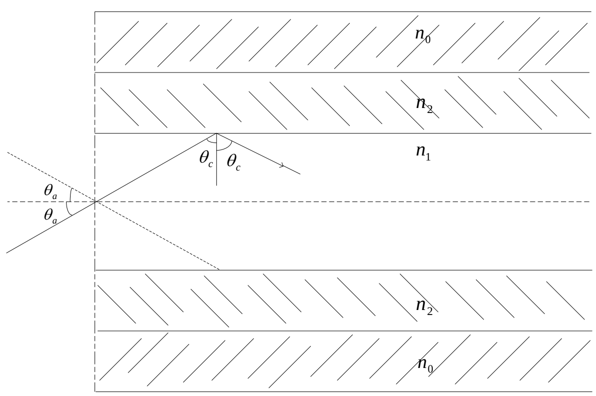

Figure 5.8: Illustrating the numerical aperture concept.

The fiber transmission efficiency is determined by the numerical aperture defined as

$$\text{NA} \equiv \sin\theta_a. \tag{5.37}$$

It can be inferred applying Snell's law (3.164) to the geometry in Fig. 5.8 that

$$n_\infty \sin\theta_a = n_1 \sin(\pi/2 - \theta_c) = n_1 \cos\theta_c \simeq \sqrt{n_1^2 - n_2^2}. \tag{5.38}$$

It then follows that in the air, where the light source is, $n_\infty = 1$, and Eq. (5.38) implies

$$\boxed{NA = \sin\theta_a = \sqrt{n_1^2 - n_2^2}}. \tag{5.39}$$

There two other dimensionless parameters that characterize optical fiber modes: the relative refractive index mismatch defined as

$$\Delta = \frac{n_1 - n_2}{n_1} \ll 1, \tag{5.40}$$

which is typically rather small, $\Delta \ll 1$, and the V-parameter,

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}. \tag{5.41}$$

The latter specifies the number of modes N trapped by the fiber:

$$\boxed{N \simeq \tfrac{1}{2}V^2}. \tag{5.42}$$

To characterize the power transmission by the fiber, we need to know a linear loss (attenuation) factor which can be determined by examining fiber losses on propagation. The power passing through the fiber at a distance L away from the source is related to the power at the source by the expression

$$P(L) = P(0)e^{-\alpha L}. \tag{5.43}$$

It is convenient to measure the attenuation constant α in dB/km, such that Eq. (5.43) can be rewritten as

$$P(L) = P(0)10^{-\alpha L/10}, \tag{5.44}$$

where the attenuation constant in dB/km is

$$\boxed{\alpha L = 10 \log_{10} \frac{P(0)}{P(L)}}. \tag{5.45}$$

In fiber optical communications one wants to avoid absorption losses at all costs. As a result, the input wave packets should have carrier frequencies far from any internal resonances of fiber core material. Under these conditions, the refractive index dependence on the frequency is fairly weak and it can be inferred from Eq. (2.21) to be

$$n^2(\omega) = 1 + \sum_{s=1}^{m} \frac{B_s \omega_s^2}{\omega_s^2 - \omega^2}. \tag{5.46}$$

Eq. (5.46) is known as the Sellmeier equation; $\{\omega_s\}$ are the resonant frequencies $\{B_s\}$ are phenomenological fitting parameters of the fiber core material.

One can then introduce the propagation constant β of a plane wave with the frequency ω by the expression

$$\boxed{\beta(\omega) = \frac{\omega}{c} n(\omega)}. \tag{5.47}$$

As $n(\omega)$ does not vary substantially with ω far from resonances, neither does β. The latter then can be expanded in a Taylor series around the carrier frequency ω_0 as

$$\boxed{\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \tfrac{1}{2}\beta_2(\omega - \omega_0)^2 + \ldots} \tag{5.48}$$

where

$$\boxed{\beta_m = \left(\frac{d^m \beta}{d\omega^m}\right)_{\omega=\omega_0}}, \qquad (m = 0, 1, 2, \ldots). \tag{5.49}$$

In particular, the first term in the expansion describes the group velocity,

$$\beta_1 = \frac{1}{v_g} = \frac{1}{c}\left(n + \omega \frac{dn}{d\omega}\right). \tag{5.50}$$

Sometimes a group refractive index n_g is also introduced such that

$$v_g = \frac{c}{n_g}, \tag{5.51}$$

where the group refractive index is given by

$$n_g = n + \omega \frac{dn}{d\omega}. \tag{5.52}$$

The second term in Eq. (5.48) is the **group velocity dispersion** parameter describing the wave packet distortion; it can be shown that

$$\beta_2 = \frac{1}{c}\left(2\frac{dn}{d\omega} + \omega \frac{d^2 n}{d\omega^2}\right). \tag{5.53}$$

Example 5. 4. Derive Eq. (5.53).
Solution. It follows from Eq. (5.49) that $\beta_2 = d\beta_1/d\omega$. Taking a derivative of Eq. (5.50) and using the rule, $(fg)' = f'g + g'f$ we arrive at Eq. (5.53).

In practice, often a different dispersion parameter D is used which is defined as

$$D \equiv \frac{d\beta_1}{d\lambda}. \tag{5.54}$$

We can show that the two dispersion parameters are related viz.,

$$\beta_2 = -\frac{D\lambda^2}{2\pi c}. \tag{5.55}$$

Example 5. 5. Derive Eq. (5.55).
Solution. It follows from Eq. (5.54) and the definition of the wavelength $\lambda = 2\pi c/\omega$ using the chain rule that

$$D = \frac{d\beta_1}{d\omega}\frac{d\omega}{d\lambda}.$$

It follows from Eq. (5.49) that the first term on the rhs is just β_2, implying that

$$D = -\frac{2\pi c \beta_2}{\lambda^2}, \Longrightarrow \beta_2 = -\frac{D\lambda^2}{2\pi c}.$$

The two dispersion parameters depend on the wavelength. Their wavelength dependence is exhibited in Fig. 5.9 for typical silica glass fibers.

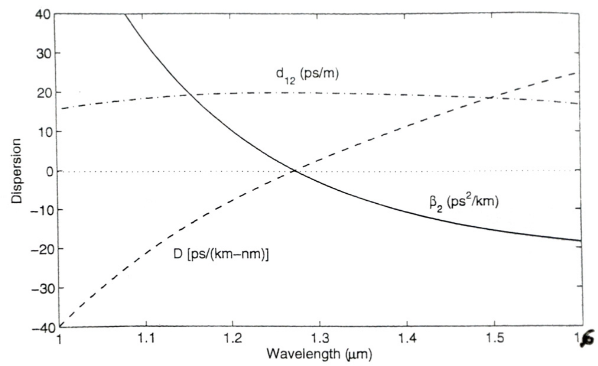

Figure 5.9: Variation of β_2, D, and d_{12} with the wavelength for fused silica. Both β_2 and D vanish at the zero dispersion wavelength corresponding to about 1.27 μm. From "Nonlinear Fiber Optics" by G. P. Agrawal.

Finally, an important phenomenon involving two closely spaced wave packets, centered at different carrier wavelengths, takes place on their propagation inside a fiber. This phenomenon is called the spatial walk-off. The spatial walk-off occurs because the two wave packets have different group velocities and the faster overtakes the slower one completely walking through it. As a result, the temporal profiles of the pulses cease to overlap, thereby drastically reducing their interactions. The spatial walk-off is characterized by the walk-off parameter

$$d_{12} = \beta_1(\lambda_1) - \beta_1(\lambda_2) = v_g^{-1}(\lambda_1) - v_g^{-1}(\lambda_2). \tag{5.56}$$

The corresponding walk-off length for pulses of typical duration T_0 is

$$L_W = T_0/|d_{12}|. \tag{5.57}$$

As an example, examine a pulse at $\lambda_1 = 1.3\mu$m co-propagating with the pulse at $\lambda_2 = 0.8\mu$m. It follows from Fig. 5.9 that $d_{12} = 20$ps/m, implying the walk-off length of just 50 cm for $T_0 = 10$ ps.