Solutions to Quiz 1, ECED 3300

Problem 1

By definition,

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial_x & \partial_y & \partial_z \\ \frac{y}{1+z^2} & \frac{-x}{1+z^2} & 0 \end{vmatrix} = -\frac{2xz}{(z^2+1)^2}\mathbf{a}_x - \frac{2yz}{(z^2+1)^2}\mathbf{a}_y - \frac{2}{z^2+1}\mathbf{a}_z$$

Problem 2

We employ the Bio-Savart law,

$$d\mathbf{H} = \frac{Id\mathbf{l} \times \mathbf{R}}{4\pi R^3}.$$

In our case, $Id\mathbf{l} = Ibd\phi\mathbf{a}_\phi$, and $\mathbf{R} = -b\mathbf{a}_\rho$; the minus sign comes from the fact that $\mathbf{R}$ is a distance from a current element to the observation point. Putting it all together,

$$d\mathbf{H} = -\frac{Ibd\phi\mathbf{a}_\phi \times \mathbf{a}_\rho b}{4\pi b^3},$$

Observing that $\mathbf{a}_\phi \times \mathbf{a}_\rho = -\mathbf{a}_z$, we obtain,

$$\mathbf{H} = \frac{I}{4\pi b}\mathbf{a}_z \int_0^{2\pi} d\phi = \left(\frac{I}{2b}\right)\mathbf{a}_z.$$

Problem 3

a) The continuity equation (charge conservation in the differential form) states,

$$\partial_t \rho_v = -\nabla \cdot \mathbf{J},$$

Thus, taking the divergence of $\mathbf{J}$, which has only a θ component, yields

$$\partial_t \rho_v = -\frac{J_0 r}{R} e^{-r/R} \left(\frac{1}{r \sin\theta}\right) \partial_\theta(\sin\theta) = -\frac{J_0 \cot\theta}{R} e^{-r/R}.$$

b) By definition,

$$I = \oint d\mathbf{S} \cdot \mathbf{J},$$

where $d\mathbf{S} = \mathbf{a}_n dS$. In our case, $\mathbf{a}_n = \mathbf{a}_r$ and hence $d\mathbf{S} \cdot \mathbf{J} = 0$, because $\mathbf{a}_r \cdot \mathbf{a}_\theta = 0$. Thus, the overall current through the surface of the sphere is equal to zero.