ECED 4310
Tutorial: Maxwell's Equations

Problem 1

Given the charge density $\rho_v(x) = xe^{-x^2}$ C/m^3, find the electric flux through the surface of the cubical box defined by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$.

Solution

Using the electric Gauss's law in the integral form, we obtain

$$\oint d\mathbf{S}\cdot\mathbf{D} = \int_v dv\rho_v = \int_0^1 dy \int_0^1 dz \int_0^1 dx x e^{-x^2} = \tfrac{1}{2}\int_0^1 d(x^2)e^{-x^2} = \tfrac{1}{2}\, e^{-x^2}\Big|_0^1 = \underline{\tfrac{1}{2}(1-e^{-1})}.$$

Problem 2

Given the electric field $\mathbf{E} = E_0 \sin\alpha x \cos(\omega t - \beta y)\mathbf{a}_z$ V/m in free space, determine $\mathbf{H}$.

Solution

The easiest way to proceed would be to use Faraday's law in the differential form. In free space,

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t}. \tag{1}$$

Taking the curl on the l.h.s., we obtain

$$\nabla\times\mathbf{E} = E_0[\mathbf{a}_x\beta\sin\alpha x\sin(\omega t-\beta y) - \mathbf{a}_y\alpha\cos\alpha x\cos(\omega t-\beta y)]. \tag{2}$$

It follows from Eqs. (1) and (2) that the magnetic field can be sought in the form

$$\mathbf{H} = \mathbf{a}_x H_x\sin\alpha x\cos(\omega t-\beta y) + \mathbf{a}_y H_y\cos\alpha x\sin(\omega t-\beta y), \tag{3}$$

where H_x and H_y are yet unknown constants. Using the Ansatz (3) on the r.h.s of Eq. (1), the latter reads

$$-\mu_0\frac{\partial\mathbf{H}}{\partial t} = \mu_0\omega[\mathbf{a}_x H_x\sin\alpha x\sin(\omega t-\beta y) - \mathbf{a}_y H_y\cos\alpha x\cos(\omega t-\beta y)]. \tag{4}$$

Equating Eqs. (2) and (4) component by component, we obtain

$$\mu_0\omega H_x = E_0\beta, \Longrightarrow, H_x = \frac{\beta E_0}{\mu_0\omega}, \tag{5}$$

and

$$\mu_0\omega H_y = E_0\alpha, \Longrightarrow, H_y = \frac{\alpha E_0}{\mu_0\omega}. \tag{6}$$

Thus,

$$\underline{\mathbf{H} = \mathbf{a}_x\frac{\beta E_0}{\mu_0\omega}\sin\alpha x\cos(\omega t-\beta y) + \mathbf{a}_y\frac{\alpha E_0}{\mu_0\omega}\cos\alpha x\sin(\omega t-\beta y).}$$

Problem 3

Given a charge distribution

$$\rho_v(x) = \begin{cases} \rho_0 x/a, & |x| \le a, \\ 0, & |x| > a, \end{cases}$$

where ρ_0 is a known constant in free space, determine the electric field everywhere.

Solution

The electric Gauss's law in differential form reads,

$$\nabla\cdot\mathbf{D} = \rho_v. \tag{7}$$

By symmetry, the electric flux density depends only on x and we assume that

$$\mathbf{D}(x) = D_x\mathbf{a}_x. \tag{8}$$

Under these conditions, the electric flux density inside the strip, $|x| \le a$ satisfies the equation

$$\frac{dD_x}{dx} = \rho_0\frac{x}{a}. \tag{9}$$

Integrating Eq. (9), we arrive at

$$D_x(x) = \frac{\rho_0 x^2}{2a} + C, \tag{10}$$

where C is a constant of integration. Outside the strip, $|x| > a$, there is no charge, and $\mathbf{D} = 0$. Since the boundaries $x = \pm a$ separate regions within the same medium, the electric flux density must be continuous across $x = \pm a$. It then follows that

$$D_x(x=\pm a) = C + \frac{\rho_0 a}{2} = 0, \Longrightarrow C = -\frac{\rho_0 a}{2}. \tag{11}$$

Thus,

$$\underline{\mathbf{D} = \begin{cases} \frac{\rho_0(x^2-a^2)}{2a}\mathbf{a}_x, & |x| \le a, \\ 0, & |x| \ge a. \end{cases}}$$

Problem 4

In free space, given the current distribution density

$$\mathbf{J} = \begin{cases} J_0(1-|z|/a)\mathbf{a}_x, & |z| \le a, \\ 0, & |z| > a, \end{cases}$$

where J_0 is a known constant, find the magnetic field everywhere.

Solution

As $\nabla \cdot \mathbf{J} = 0$, it follows from the continuity equation that $\partial_t \rho_v = 0$, implying that the charge density is time-independent. It can then be inferred from the electric Gauss's law (7) that $\mathbf{D}$ is time-independent as well. The Ampère's law then states that

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{12}$$

First, of all, outside the strip, $|z| \ge a$, there is no current, and hence no magnetic field, $\mathbf{H} = 0$. Inside the strip, we assume by symmetry that (i) $\mathbf{H}$ may only depend on z and (ii) it should have no x-component. We suppose that

$$\mathbf{H} = H(z)\mathbf{a}_y, \tag{13}$$

which manifestly satisfies $\nabla \cdot \mathbf{H} = 0$. Taking the curl on the l.h.s of Eq. (12), we obtain $\nabla \times \mathbf{H} = -\partial_z H \mathbf{a}_x$, implying that

$$-\frac{dH}{dz} = J_0(1-|z|/a). \tag{14}$$

Integrating Eq. (14), we obtain

$$H(z) = \begin{cases} -J_0(z - z^2/2a) + C_1, & 0 < z \le a, \\ -J_0(z + z^2/2a) + C_2, & -a \le z < 0, \end{cases} \tag{15}$$

where $C_{1,2}$ are unknown constants. The continuity of the magnetic field across the boundary $z = a$ implies that

$$-J_0(a - a/2) + C_1 = 0, \Longrightarrow C_1 = J_0 a/2.$$

By the same token, the continuity of $\mathbf{H}$ across $z = -a$ leads to

$$-J_0(-a + a/2) + C_2 = 0, \Longrightarrow C_2 = -J_0 a/2.$$

Consequently,

$$\mathbf{H} = \begin{cases} -J_0\left(z - \frac{z^2+a^2}{2a}\right)\mathbf{a}_y, & 0 < z \le a \\ -J_0\left(z + \frac{z^2+a^2}{2a}\right)\mathbf{a}_y, & -a \le z < 0, \\ 0 & |z| \ge a. \end{cases}$$

or in a more compact form

$$\underline{\mathbf{H} = \begin{cases} -J_0\left[z - \frac{|z|(z^2+a^2)}{2az}\right]\mathbf{a}_y, & |z| \le a, \\ 0 & |z| \ge a. \end{cases}}$$

Problem 5

A rod of length l rotates about the z-axis with the angular velocity ω. If $\mathbf{B} = B_0\mathbf{a}_z$, determine the voltage induced in the rod.

Solution

Assume the rod was located along the x-axis at $t = 0$. It follows that at the time t, it makes the angle $\phi = \omega t$ with the x-axis. We apply Faraday's law in the integral form to the sector formed with the x-axis and the position of the rod at the time t.

$$\mathcal{E}_{\text{emf}} = -\frac{d}{dt}\int d\mathbf{S}\cdot\mathbf{B} = -\frac{d}{dt}B_0\left(\mathbf{a}_z\cdot\mathbf{a}_z\tfrac{1}{2}l^2\phi\right) = -\tfrac{1}{2}B_0 l^2\tfrac{d}{dt}\omega t = -\tfrac{1}{2}B_0 l^2\omega.$$

Thus,

$$V = \mathcal{E}_{\text{emf}} = \underline{\tfrac{1}{2}B_0\omega l^2}.$$