ECED 4310
Tutorial: Boundary conditions to Maxwell's Equations

Problem 1

Show that neither electric nor magnetic time-varying field can exist within a perfect conductor, $\sigma \to \infty$. What are the boundary conditions for a perfect conductor?

Solution

It follows at once from Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$ that for the conduction current in the prefect conductor, $\sigma \to \infty$, to be finite, $\mathbf{E} = 0$ inside the conductor. It then follows from Faraday's law that $\mathbf{B} = 0$ as well. Moreover, the electric Gauss's law implies that $\rho_v = 0$ in the interior of the perfect conductor. The boundary conditions at the surface of a perfect conductor read then

$$\mathbf{a}_n \times \mathbf{E} = 0, \tag{1}$$

$$\mathbf{a}_n \cdot \mathbf{D} = \rho_s, \tag{2}$$

$$\mathbf{a}_n \times \mathbf{H} = \mathbf{J}_s, \tag{3}$$

$$\mathbf{a}_n \cdot \mathbf{B} = 0. \tag{4}$$

Here $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{B}$ are the fields on the surface of the perfect conductor; ρ_s and $\mathbf{J}_s$ are the surface charge and current density on the conductor, and $\mathbf{a}_n$ is an outward unit normal.

Problem 2

The interface $x = 0$ separates two perfect dielectric media 1 and 2, occupying the half-spaces $x > 0$ and $x < 0$, respectively. The media properties are: $\epsilon_1 = 12\epsilon_0$, $\mu_1 = 2\mu_0$, $\epsilon_2 = 9\epsilon_0$, and $\mu_2 = \mu_0$. Given $\mathbf{E}_1 = E_0(3\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z)$ and $\mathbf{H}_1 = H_0(2\mathbf{a}_x - 3\mathbf{a}_y)$, determine $\mathbf{E}_2$ and $\mathbf{H}_2$.

Solution

In this problem, the unit normal to the interface is $\mathbf{a}_n = -\mathbf{a}_x$. Thus, $\mathbf{E}_{1t} = E_0(2\mathbf{a}_y - 6\mathbf{a}_z)$ and $\mathbf{E}_{1n} = 3E_0\mathbf{a}_x$. It follows by continuity of tangential components of $\mathbf{E}$ that $\mathbf{E}_{2t} = \mathbf{E}_{1t} = E_0(2\mathbf{a}_y - 6\mathbf{a}_z)$. Next, $\mathbf{D}_1 = 12E_0\epsilon_0(3\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z)$. Let, $\mathbf{D}_2 = 9E_0\epsilon_0(\alpha\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z)$. It follows from the boundary conditions that

$$\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = 0, \Longrightarrow (36 - 9\alpha)E_0\epsilon_0 = 0, \Longrightarrow \alpha = 4.$$

Thus,

$$\underline{\mathbf{E}_2 = E_0(4\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z)},$$

By the same token, the continuity of tangential components of $\mathbf{H}$ in the absence of surface currents, implies that $\mathbf{H}_{2t} = \mathbf{H}_{1t} = -3H_0\mathbf{a}_y$. Further, $\mathbf{B}_1 = 2\mu_0 H_0(2\mathbf{a}_x - 3\mathbf{a}_y)$, and $\mathbf{B}_2 = \mu_0 H_0(\beta\mathbf{a}_x - 3\mathbf{a}_y)$. The continuity of normal components of the flux density leads to

$$\mu_0 H_0(\beta - 4) = 0, \Longrightarrow \beta = 4.$$

Thus,

$$\underline{\mathbf{H}_2 = H_0(4\mathbf{a}_x - 3\mathbf{a}_y)},$$

Problem 3

The electric flux density at a point on the surface of a perfect conductor is given by $\mathbf{D} = D_0(\mathbf{a}_x + \sqrt{3}\mathbf{a}_y + 2\sqrt{3}\mathbf{a}_z)$. Find the magnitude of the surface charge density at the point.

Solution

The boundary conditions on the surface of a perfect conductor state,

$$\mathbf{a}_n \times \mathbf{D} = 0, \tag{5}$$

and

$$\mathbf{a}_n \cdot \mathbf{D} = \rho_s. \tag{6}$$

The only way to satisfy Eq. (5) is for the unit normal to the conductor at the point to be parallel $\mathbf{D}$, i.e.,

$$\mathbf{a}_n = \pm s(\mathbf{a}_x + \sqrt{3}\mathbf{a}_y + 2\sqrt{3}\mathbf{a}_z), \tag{7}$$

where s is a normalization constant, obtained from the condition, $|\mathbf{a}_n| = 1$, $s = 1/4$. Thus,

$$\mathbf{a}_n = \pm\frac{(\mathbf{a}_x + \sqrt{3}\mathbf{a}_y + 2\sqrt{3}\mathbf{a}_z)}{4}, \tag{8}$$

It follows from Eqs. (6) and (8) that

$$\underline{|\rho_s| = 4|D_0|}. \tag{9}$$

Problem 4

At a point on the surface of a perfect conductor, $\mathbf{D} = D_0(\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)$ and $\mathbf{H} = H_0(2\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z)$. It is known that $\rho_s > 0$. Determine ρ_S and $\mathbf{J}_s$ at the point.

Solution

It follows from Eq. (5) that in this case,

$$\mathbf{a}_n = \frac{(\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)}{3}. \tag{10}$$

We chose a "+" sign because ρ_s is positive. Indeed, it follows from Eqs. (6) and (refaux5) that

$$\underline{\rho_s = \mathbf{a}_n \cdot \mathbf{D} = 3D_0 > 0}. \tag{11}$$

The other boundary condition implies that

$$\mathbf{J}_s = \mathbf{a}_n \times \mathbf{H}, \tag{12}$$

or

$$\mathbf{J}_s = \frac{H_0}{3} \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{vmatrix}. \tag{13}$$

Working out the determinant, we obtain

$$\underline{\mathbf{J}_s = H_0(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)}.$$