ECED 3300, Fall 2018
Electromagnetic Fields
Solutions to the Midterm Examination Problems

Problem 1

a) Using Gauss's law in the differential form,

$$\rho_v = \nabla \cdot \mathbf{D} = \tfrac{1}{\rho}\partial_\rho(\rho D_\rho) + \tfrac{1}{\rho}\partial_\phi D_\phi + \partial_z D_z = \tfrac{1}{\rho}\partial_\rho(\rho \tfrac{1}{\rho} e^{-\rho^2})\cos\phi + e^{-\rho^2}\partial_\phi(\sin\phi) = -e^{-\rho^2}\cos\phi.$$

b) $\nabla \times \mathbf{E} = 0$. Alternatively, there exists a scalar field V—electrostatic potential—such that $\mathbf{E} = -\nabla V$.

Problem 2

a) Using Gauss's law in the integral form for a spherical surface which is a Gaussian surface due to the spherical symmetry of the charge, we identify two regions of interest

- *Cloud interior,* $r \leq a$

$$D 4\pi r^2 = \rho_v 4\pi r^3/3, \implies \mathbf{D} = \mathbf{a}_r \rho_v r/3,$$

- *Cloud exterior,* $r \geq a$,

$$D 4\pi r^2 = \rho_v 4\pi a^3/3, \implies \mathbf{D} = \mathbf{a}_r \rho_v a^3/3r^2.$$

Since in free space $\mathbf{D} = \epsilon_0 \mathbf{E}$, it follows that

$$\mathbf{E} = \begin{cases} \left(\frac{\rho_v r}{3\epsilon_0}\right)\mathbf{a}_r, & r \leq a; \\ \left(\frac{\rho_v a^3}{3\epsilon_0 r^2}\right)\mathbf{a}_r, & r \geq a. \end{cases}$$

b) By definition,

$$V_0 = -\int_\infty^0 d\mathbf{l}\cdot\mathbf{E} = -\int_\infty^a dr E_>(r) - \int_a^0 dr E_<(r).$$

Thus,

$$V_0 = -\int_\infty^a dr \frac{\rho_v a^3}{3\epsilon_0 r^2} - \int_a^0 dr \frac{\rho_v r}{3\epsilon_0}.$$

Hence,

$$V_0 = \frac{\rho_v a^3}{3\epsilon_0}\frac{1}{r}\bigg|_\infty^a - \frac{\rho_v}{3\epsilon_0}\frac{r^2}{2}\bigg|_a^0 = \frac{\rho_v a^2}{2\epsilon_0}.$$

c) By definition,

$$W = qV_0 = \frac{q\rho_v a^2}{2\epsilon_0}.$$

Problem 3

We've got a series connection of two spherical capacitors with capacitances,

$$C_1 = \frac{4\pi\epsilon_1}{1/R_1 - 1/a}, \qquad C_2 = \frac{4\pi\epsilon_2}{1/a - 1/R_2}.$$

Hence,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{(1/R_1 - 1/a)}{4\pi\epsilon_1} + \frac{(1/a - 1/R_2)}{4\pi\epsilon_2}.$$

The energy can be calculated using the expression,

$$W_E = \frac{Q^2}{2C} = \frac{Q^2}{8\pi}\left(\frac{1/R_1 - 1/a}{\epsilon_1} + \frac{1/a - 1/R_2}{\epsilon_2}\right).$$

Problem 4

a) There are two images: $-Q_1$, placed at $(-a, o, -h)$ and $-Q_2$, placed at $(a, 0, -h)$. The locations are chosen to ensure that the overall potential is equal to zero on the plane. Justification,

$$\begin{aligned} V \;=\; & \frac{Q_1}{4\pi\epsilon_0\sqrt{(x+a)^2 + y^2 + (z-h)^2}} + \frac{Q_2}{4\pi\epsilon_0\sqrt{(x-a)^2 + y^2 + (z-h)^2}} \\ & - \frac{Q_1}{4\pi\epsilon_0\sqrt{(x+a)^2 + y^2 + (z+h)^2}} - \frac{Q_2}{4\pi\epsilon_0\sqrt{(x-a)^2 + y^2 + (z+h)^2}}. \end{aligned} \tag{1}$$

Check that

$$V|_z = 0 = 0.$$

b) By Coulomb's law,

$$\mathbf{F}_1 = -\frac{Q_1^2}{4\pi\epsilon_0(2h)^2}\mathbf{a}_z + \frac{Q_1Q_2}{4\pi\epsilon_0(2a)^2}\mathbf{a}_x - \frac{Q_1Q_2}{4\pi\epsilon_0(4a^2+4h^2)}\mathbf{a}_s,$$

where

$$\mathbf{a}_s = \mathbf{a}_x\cos\alpha - \mathbf{a}_z\sin\alpha, \qquad \tan\alpha = h/a.$$

It follows that

$$\mathbf{F}_1 = -\frac{Q_1^2}{4\pi\epsilon_0(2h)^2}\mathbf{a}_z + \frac{Q_1Q_2}{4\pi\epsilon_0(2a)^2}\mathbf{a}_x - \frac{Q_1Q_2}{4\pi\epsilon_0(4a^2+4h^2)}\mathbf{a}_s,$$

Simplifying,

$$\mathbf{F}_1 = \frac{Q_1}{16\pi\epsilon_0}\left[-\frac{Q_1}{h^2}\mathbf{a}_z + \frac{Q_2}{a^2}\mathbf{a}_x - \frac{Q_2}{(a^2+h^2)}\mathbf{a}_s\right],$$

where

$$\mathbf{a}_s = \frac{a\mathbf{a}_x - h\mathbf{a}_z}{\sqrt{a^2+h^2}}.$$

c) By energy conservation,

$$W = W_E^{(f)} - W_E^{(i)}.$$

Here,

$$W_E^{(i)} = \tfrac{1}{2}Q_1V_1^i + \tfrac{1}{2}Q_2V_2^i + \tfrac{1}{2}(-Q_1)\times 0 + \tfrac{1}{2}(-Q_2)\times 0. \tag{2}$$

and since $V_2^f = 0$ (far away point),

$$W_E^{(f)} = \tfrac{1}{2}Q_1V_1^f + \tfrac{1}{2}(-Q_1)\times 0 + \tfrac{1}{2}(-Q_2)\times 0. \tag{3}$$

Potential is always zero behind the plane. Above the plane, we obtain by the superposition principle

$$V_1^i = -\frac{Q_1}{4\pi\epsilon_0(2h)} + \frac{Q_2}{4\pi\epsilon_0(2a)} - \frac{Q_2}{4\pi\epsilon_0\sqrt{4a^2+4h^2}}, \tag{4}$$

and

$$V_2^i = -\frac{Q_2}{4\pi\epsilon_0(2h)} + \frac{Q_1}{4\pi\epsilon_0(2a)} - \frac{Q_1}{4\pi\epsilon_0\sqrt{4a^2+4h^2}}. \tag{5}$$

By the same token,

$$V_1^f = -\frac{Q_1}{4\pi\epsilon_0(2h)}. \tag{6}$$

It follows from Eqs. (2), (4) and (5) that

$$W_E^{(i)} = -\frac{Q_1^2}{16\pi\epsilon_0 h} - \frac{Q_2^2}{16\pi\epsilon_0 h} + \frac{Q_1Q_2}{8\pi\epsilon_0 a} - \frac{Q_1Q_2}{8\pi\epsilon_0\sqrt{a^2+h^2}}. \tag{7}$$

and

$$W_E^{(f)} = -\frac{Q_1^2}{16\pi\epsilon_0 h}. \tag{8}$$

Thus, Eqs. (7) and (8) imply that

$$W = W_E^{(f)} - W_E^{(i)} = \frac{Q_2^2}{16\pi\epsilon_0 h} - \frac{Q_1Q_2}{8\pi\epsilon_0 a} + \frac{Q_1Q_2}{8\pi\epsilon_0\sqrt{a^2+h^2}}. \tag{9}$$