ECED 3300, Fall 2019
Electromagnetic Fields
Solutions to the Midterm Examination Problems

Problem 1

1. It follows from Poisson's equation that in free space,

$$\nabla^2 V = -\rho_v/\epsilon_0, \implies \rho_v = -\epsilon_0 \nabla^2 V.$$

In our case, $V = V(\rho, \phi)$, hence in the cylindrical coordinates,

$$\nabla^2 V = \frac{1}{\rho}\,\partial_\rho(\rho\,\partial_\rho V) + \frac{1}{\rho^2}\partial^2_{\phi\phi} V,$$

It follows that

$$\nabla^2 V = \frac{1}{\rho}\partial_\rho(\rho\cos\phi) - \frac{1}{\rho}\cos\phi = \cos\phi/\rho - \cos\phi/\rho = 0.$$

Hence,

$$\underline{\rho_v = 0.}$$

in the given region of space.

2.

$$\mathbf{E} = -\nabla V = -\mathbf{a}_\rho \partial_\rho V - \frac{\mathbf{a}_\phi}{\rho}\partial_\phi V = -\mathbf{a}_\rho \cos\phi + \mathbf{a}_\phi \sin\rho$$

It follows at once that

$$|\mathbf{E}|^2 = 1.$$

The energy is given by the volume integral,

$$W_E = \frac{\epsilon_0}{2}\int dv|\mathbf{E}|^2 = \frac{\epsilon_0}{2}\int_0^\pi d\phi\int_0^2 d\rho\,\rho\int_0^1 dz = \frac{\epsilon_0}{2}\times\pi\times\left.\frac{\rho^2}{2}\right|_0^2.$$

Hence,

$$W_E = \frac{\epsilon_0}{2}\times\pi\times\frac{2^2}{2} = \underline{\pi\epsilon_0}.$$

Problem 2

Assuming spherical symmetry, $\mathbf{D} = D\mathbf{a}_r$, we apply Gauss's law to the interior and exterior of the sphere separately.

a) Interior, $r < R$:

$$D\,4\pi r^2 = \rho_0\frac{4\pi}{3}r^3 \implies D = \rho_0 r/3,$$

implying that

$$\mathbf{D} = \frac{\rho_0 r}{3}\mathbf{a}_r \Longrightarrow \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0\epsilon_r} = \frac{\rho_0 r}{3\epsilon_0\epsilon_r}\mathbf{a}_r.$$

b) Exterior, $r > R$:

$$D\,4\pi r^2 = \rho_0\frac{4\pi}{3}R^3 \Longrightarrow D = \rho_0 R^3/3r^2,$$

implying that

$$\mathbf{D} = \frac{\rho_0 R^3}{3r^2}\mathbf{a}_r \Longrightarrow \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\rho_0 R^3}{3\epsilon_0 r^2}\mathbf{a}_r.$$

By definition,

$$V = -\int_\infty^R d\mathbf{l}\cdot\mathbf{E} - \int_R^0 d\mathbf{l}\cdot\mathbf{E}.$$

Since $\mathbf{E} = E\mathbf{a}_r$, $d\mathbf{l}\cdot\mathbf{E} = dr E$. It then follows that

$$V = -\int_\infty^R dr\,\frac{\rho_0 R^3}{3\epsilon_0 r^2} - \int_R^0 dr\,\frac{\rho_0 r}{3\epsilon_0\epsilon_r}.$$

Simplifying,

$$V = -\frac{\rho_0}{3\epsilon_0}\left[-\int_\infty^R dr\,\frac{R^3}{r^2} - \frac{1}{\epsilon_r}\int_R^0 dr\,r\right] = \frac{\rho_0}{3\epsilon_0}\left[\left.\frac{R^3}{r}\right|_\infty^R - \left.\frac{r^2}{2\epsilon_r}\right|_R^0\right]$$

Using the Newton-Leibniz formula, we arrive at

$$V = \frac{\rho_0}{3\epsilon_0}\left(R^2 + \frac{R^2}{2\epsilon_r}\right) = \underline{\frac{\rho_0 R^2}{6\epsilon_0\epsilon_r}(1+2\epsilon_r).}$$

Problem 3

1) Working in the Cartesian coordinates, we obtain in the upper half-space,

$$\mathbf{E} = -\nabla V = -\mathbf{a}_z\partial_z V = \mathbf{a}_z(V_0/a)e^{-z/a}.$$

Since the lower half-space is filled with a perfect conductor, $\mathbf{E} = 0$. Combining,

$$\mathbf{E} = \begin{cases} \mathbf{a}_z(V_0/a)e^{-z/a} & z > 0 \\ 0 & z < 0 \end{cases}$$

2) Using the boundary conditions at a conductor-dielectric interface.

$$(\mathbf{D}_1 - \mathbf{D}_2)|_{z=0}\cdot\mathbf{a}_{n21} = \rho_s$$

Medium "1" is the upper half-space, $z > 0$ and medium "2" is the lower half-space, $z < 0$. Hence, $\mathbf{a}_{n21} = \mathbf{a}_z$. It follows that

$$\mathbf{D}_1|_{z=0}\cdot\mathbf{a}_{n21} = D_z = \epsilon E_z|_{z=0} = \epsilon V_0/a.$$

Also,

$$\mathbf{D}_2 = 0,$$

as the flux density inside the conductor. We can then infer that

$$\underline{\rho_s = \epsilon V_0/a.}$$

Problem 4

Starting with the superposition principle for the potential,

$$V = \int \frac{dl\,\rho_l}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}, \tag{1}$$

we adopt the cylindrical coordinates with the origin at the ring center. In this case, the observation point is at the origin. Hence, $\mathbf{r} = 0$. Further, $\mathbf{r}' = b\mathbf{a}_\rho$, implying that $|\mathbf{r}-\mathbf{r}'| = b$. As well, we choose an infinitesimally small arc on the ring subtending the angle $d\phi$ as viewed from the center. The corresponding infinitesimal charge reads,

$$dl\,\rho_l = bd\phi\,\rho_0\cos^2\phi. \tag{2}$$

On substituting from Eq. 2 into Eq. 1, and using the fact that $|\mathbf{r}-\mathbf{r}'| = b$, we obtain

$$V = \frac{\rho_0}{4\pi\epsilon_0}\int_0^{2\pi}\frac{d\phi\,b\cos^2\phi}{b} = \frac{\rho_0}{4\pi\epsilon_0}\int_0^{2\pi}d\phi\,\cos^2\phi \tag{3}$$

Looking up the table integral from the formula sheet, we arrive at

$$V = \frac{\rho_0}{4\pi\epsilon_0}\left(\frac{\phi}{2}+\frac{\cos 2\phi}{4}\right)\Bigg|_0^{2\pi} = \underline{\frac{\rho_0}{4\epsilon_0}}. \tag{4}$$