Midterm Solutions, ECED 3300

Problem 1.

$$\mathbf{D}(\rho,\phi,z) = \underbrace{e^{-|z|}\sin\phi}_{D_\rho}\,\mathbf{a}_\rho + \underbrace{e^{-|z|}\cos\phi}_{D_\phi}\,\mathbf{a}_\phi + \underbrace{e^{-\rho^2-z^2}}_{D_z}\mathbf{a}_z.$$

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{1}{\rho}\frac{\partial(\rho D_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}.$$

Thus,

$$\rho_v = -2ze^{-z^2-\rho^2}.$$

Problem 2. a) Choosing $-Q$ placed symmetrically with respect to the plane. Justification: the potential distribution in the upper half-space $z > 0$,

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z-h)^2}} - \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z+h)^2}},$$

satisfies the boundary condition $V|_{z=0} = 0$.

(b)

$$\mathbf{F} = -\frac{Q^2}{4\pi\epsilon_0(2h)^2}\mathbf{a}_z = -\frac{Q^2}{16\pi\epsilon_0 h^2}\mathbf{a}_z.$$

(c)

$$W_E = \tfrac{1}{2}QV_Q = \tfrac{1}{2}Q\left(-\tfrac{Q}{4\pi\epsilon_0 2h}\right) = -\tfrac{Q^2}{16\pi\epsilon_0 h}.$$

(d) Energy conservation implies that

$$W = W_E^{(f)} - W_E^{(i)}.$$

Next, from part (c),

$$W_E^{(i)} = -\frac{Q^2}{16\pi\epsilon_0 h}.$$

Further,

$$W_E^{(f)} = \tfrac{1}{2}qV_q + \tfrac{1}{2}QV_Q.$$

To satisfy the boundary condition at the interface at all time, we would need four charges: the original charge Q and its image, $-Q$, the probe charge q and its image, $-q$. Therefore,

$$V_q = \frac{Q}{4\pi\epsilon_0\sqrt{a^2+a^2+(b-h)^2}} - \frac{q}{4\pi\epsilon_0(2b)} - \frac{Q}{4\pi\epsilon_0\sqrt{a^2+a^2+(b+h)^2}},$$

and

$$V_Q = \frac{q}{4\pi\epsilon_0\sqrt{a^2+a^2+(h-b)^2}} - \frac{Q}{4\pi\epsilon_0(2h)} - \frac{q}{4\pi\epsilon_0\sqrt{a^2+a^2+(h+b)^2}}.$$

Finally,

$$W = \frac{qQ}{4\pi\epsilon_0\sqrt{2a^2 + (b-h)^2}} - \frac{qQ}{4\pi\epsilon_0\sqrt{2a^2 + (b+h)^2}} - \frac{q^2}{16\pi\epsilon_0 b}.$$

Problem 3. a)

$$\nabla^2 V = -\rho_v/\epsilon,$$

The charge distribution symmetry dictates that

$$\frac{d^2V}{dz^2} = -\frac{\rho_0}{\epsilon}e^{-z/a}.$$

Integrating twice, we obtain,

$$V(z) = C_1 z + C_2 - \frac{\rho_0 a^2}{\epsilon}e^{-z/a}.$$

The “natural” boundary condition, $V \to 0$ as $z \to +\infty$ implies that $C_1 = C_2 = 0$. Thus,

$$V(z) = -\frac{\rho_0 a^2}{\epsilon}e^{-z/a}.$$

b)

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial z}\mathbf{a}_z = -\mathbf{a}_z\frac{\rho_0 a}{\epsilon}e^{-z/a}.$$

in the upper half-space, $z > 0$ and $\mathbf{E} = 0$ (conductor) in the lower half-space, $z < 0$. Thus,

$$\mathbf{E} = \begin{cases} -\mathbf{a}_z\frac{\rho_0 a}{\epsilon}e^{-z/a}, & z > 0 \\ 0, & z < 0 \end{cases}$$

c) The boundary condition at the interface $z = 0$ implies that

$$\rho_s = (\mathbf{D}_1 - \mathbf{D}_2)|_{z=0} \cdot \mathbf{a}_{n_{21}}.$$

Here, $\mathbf{D}_2 = 0$, $\mathbf{D}_1 = -\mathbf{a}_z\rho_0 a e^{-z/a}$, and $\mathbf{a}_{n_{21}} = \mathbf{a}_z$. Finally,

$$\rho_s = -\rho_0 a.$$

Problem 4. We’ve got a series connection of two spherical capacitors with capacitances,

$$C_1 = \frac{4\pi\epsilon_1}{1/a - 1/c}, \qquad C_2 = \frac{4\pi\epsilon_2}{1/c - 1/b}.$$

Hence,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}.$$

Finally,

$$C = \frac{4\pi\epsilon_1\epsilon_2}{\epsilon_2(1/a - 1/c) + \epsilon_1(1/c - 1/b)}.$$