Prob. 6.15

$$\frac{1}{\rho}\frac{d^2V}{d\phi^2}=0 \quad\longrightarrow\quad \frac{d^2V}{d\phi^2}=0 \quad\longrightarrow\quad \frac{dV}{d\phi}=A$$

$$V = A\phi + B$$

$$0=0+B \quad\longrightarrow\quad B=0$$

$$50 = A\pi/2 \quad\longrightarrow\quad A=\frac{100}{\pi}$$

$$E=-\nabla V=-\frac{1}{\rho}\frac{dV}{d\phi}\boldsymbol{a}_\phi=-\frac{A}{\rho}\boldsymbol{a}_\phi=\underline{\underline{-\frac{100}{\pi\rho}\boldsymbol{a}_\phi}}$$

Prob. 6.16

(a)

$$\nabla^2 V=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+0$$

$$=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(2c_1\rho^2-2c_2\rho^{-2}\right)\sin 2\phi-\frac{4}{\rho^2}(c_1\rho^2+c_2\rho^{-2})\sin 2\phi$$

$$=(4c_1+4c_2\rho^{-4}-4c_1-4c_2\rho^{-4})\sin 2\phi=0$$

(b)

At $P(2,45^\circ,1), \rho=1, \phi=45^o$

$$50=(c_1+c_2)\sin 90^o=c_1+c_2 \qquad (1)$$

$$-E=\nabla V=\frac{\partial V}{\partial\rho}\boldsymbol{a}_\rho+\frac{1}{\rho}\frac{\partial V}{\partial\phi}\boldsymbol{a}_\phi+\boldsymbol{0}$$

$$=(2c_1\rho-2c_2\rho^{-3})\sin 2\phi\boldsymbol{a}_\rho+(c_1\rho+c_2\rho^{-3})(2)\cos 2\phi\boldsymbol{a}_\phi$$

At P,

$$-E=(2c_1-2c_2)(1)\boldsymbol{a}_\rho+\boldsymbol{0}$$

$$|\mathrm{E}|=100=2c_1-2c_2 \quad\longrightarrow\quad 50=c_1-c_2 \qquad (2)$$

From (1) and (2), $\underline{\underline{c_1=50, c_2=0}}$

or

$$F'' + \cot\theta F' + \lambda F = 0$$

Also,

$$\frac{d}{dr}(r^2 R') - \lambda R = 0$$

or

$$R'' + \frac{2R'}{r} - \frac{\lambda}{r^2} R = 0$$

Prob. 6.29 If the centers at $\phi = 0$ and $\phi = \pi / 2$ are maintained at a potential difference of V_o, from Example 6.3,

$$E_\phi = \frac{2V_o}{\pi\rho}, \qquad J = \sigma E$$

Hence,

$$I = \int J \bullet dS = \frac{2V_o\sigma}{\pi} \int_{\rho=a}^{b} \int_{z=0}^{t} \frac{1}{\rho} d\rho dz = \frac{2V_o\sigma t}{\pi} \ln(b/a)$$

and

$$R = \frac{V_o}{I} = \frac{\pi}{2\sigma t \ln(b/a)}$$

Prob. 6.30 If $V(r=a) = 0, \quad V(r=b) = V_o$, from Example 6.9,

$$E = \frac{V_o}{r^2(1/a - 1/b)}, \qquad J = \sigma E$$

Hence,

$$I = \int J \bullet dS = \frac{V_o\sigma}{1/a - 1/b} \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} \frac{1}{r^2} r^2 \sin\theta d\theta d\phi = \frac{2\pi V_o\sigma}{1/a - 1/b} (-\cos\theta) |_0^\alpha$$

$$R = \frac{V_o}{I} = \frac{\frac{1}{a} - \frac{1}{b}}{2\pi\sigma(1 - \cos\alpha)}$$

Prob. 6.31

This is the same as Problem 6.30 except that $\alpha = \pi$. Hence,

$$R = \frac{1}{2\pi\sigma(1-\cos\pi)}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{4\pi\sigma}\left(\frac{1}{a}-\frac{1}{b}\right)$$

Prob. 6.32 For a spherical capacitor, from Eq. (6.38),

$$R = \frac{\frac{1}{a}-\frac{1}{b}}{4\pi\sigma}$$

For the hemisphere, $R' = 2R$ since the sphere consists of two hemispheres in parallel. As

$b \longrightarrow \infty$,

$$R' = \lim_{b\longrightarrow\infty} \frac{2\left[\frac{1}{a}-\frac{1}{b}\right]}{4\pi\sigma} = \frac{1}{2\pi a\sigma}$$

$$G = 1/R' = 2\pi a\sigma$$

Alternatively, for an isolated sphere, $C = 4\pi\varepsilon a$. But

$$RC = \frac{\varepsilon}{\sigma} \longrightarrow R = \frac{1}{4\pi a\sigma}$$

$$R' = 2R = \frac{1}{2\pi a\sigma} \quad \text{or} \quad G = 2\pi a\sigma$$

Prob. 6.33

(a) For the parallel-plate capacitor,

$$\mathbf{E} = -\frac{V_o}{d}\mathbf{a}_x$$

From Example 6.11,

$$C = \frac{1}{V_o^2}\int \varepsilon\,|\,\mathbf{E}\,|^2 dv = \frac{1}{V_o^2}\int \varepsilon\frac{V_o^2}{d^2}dv = \frac{\varepsilon}{d^2}Sd = \frac{\varepsilon S}{d}$$

(b) For the cylindrical capacitor,

$$\mathbf{E} = -\frac{V_o}{\rho \ln b/a}\mathbf{a}_\rho$$

Prob. 6.51

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)} = \frac{2\pi \times \dfrac{10^{-9}}{36\pi} \times 4.2 \times 400 \times 10^{-3}}{\ln(3.5/1)} = \underline{\underline{74.5 \text{ pF}}}$$

Prob. 6.52

$$C = \frac{2\pi\varepsilon_o L}{\ln(b/a)} = \frac{2\pi \times \dfrac{10^{-9}}{36\pi} \times 100 \times 10^{-6}}{\ln(600/20)} = 1.633 \times 10^{-15} \text{ F}$$

$$V = Q/C = \frac{50 \times 10^{-15}}{1.633 \times 10^{-15}} = \underline{\underline{30.62 \text{ V}}}$$

Prob. 6.53

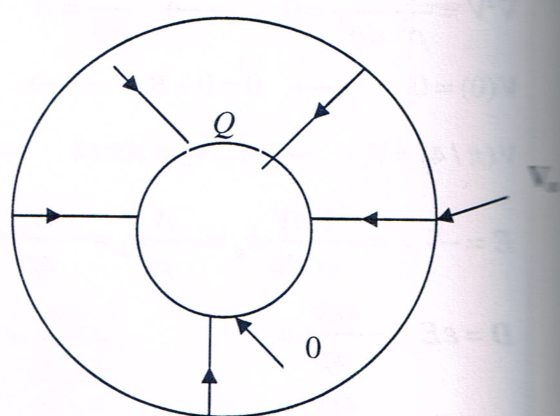
$$C_1 = \frac{2\pi\varepsilon_1}{\ln(b/a)}, \quad C_2 = \frac{2\pi\varepsilon_2}{\ln(c/b)}$$

Since the capacitance are in series, the total capacitance per unit length is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \underline{\underline{\frac{2\pi\varepsilon_1\varepsilon_2}{\varepsilon_2 \ln(b/a) + \varepsilon_1 \ln(c/b)}}}$$

Prob. 6.54

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon r^2}\boldsymbol{a}_r$$



$$W = \frac{1}{2}\int \varepsilon |\boldsymbol{E}|^2 dv = \iiint \frac{Q^2}{32\pi^2\varepsilon^2 r^4}\varepsilon r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{Q^2}{32\pi^2\varepsilon}(2\pi)(2)\int_c^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$W = \frac{Q^2(b-c)}{8\pi\varepsilon bc}$$

Prob. 6.55

(a) Method 1: $\boldsymbol{E} = \frac{\rho_s}{\varepsilon}(-\boldsymbol{a}_x)$, where ρ_s is to be determined.

$$V_o = -\int \boldsymbol{E} \bullet dl = -\int \frac{-\rho_s}{\varepsilon} dx = \rho_s \int_0^d \frac{1}{\varepsilon_o} \frac{d}{d+x} dx = \frac{\rho_s}{\varepsilon} d \ln(x+d) \Big|_0^d$$

$$V_o = \rho_s d \ln\frac{2d}{d} \longrightarrow \rho_s = \frac{V_o \varepsilon_o}{d \ln 2}$$

$$\boldsymbol{E} = -\frac{\rho_s}{\varepsilon}\boldsymbol{a}_x = -\frac{V_o}{(x+d)\ln 2}\boldsymbol{a}_x$$

Method 2: We solve Laplace's equation

$$\nabla \bullet (\varepsilon \nabla V) = \frac{d}{dx}\left(\varepsilon \frac{dV}{dx}\right) = 0 \longrightarrow \varepsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\varepsilon} = \frac{Ad}{\varepsilon_o (x+d)} = \frac{c_1}{x+d}$$

$$V = c_1 \ln(x+d) + c_2$$

$$V(x=0) = 0 \longrightarrow 0 = c_1 \ln d + c_2 \longrightarrow c_2 = -c_1 \ln d$$

$$V(x=d) = V_o \longrightarrow V_o = c_1 \ln 2d - c_1 \ln d = c_1 \ln 2$$

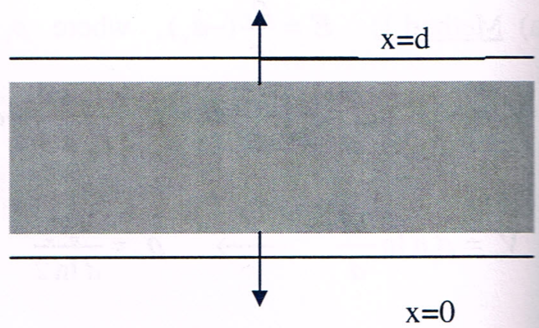
$$c_1 = \frac{V_o}{\ln 2}$$

$$V = c_1 \ln\frac{x+d}{d} = \frac{V_o}{\ln 2} \ln\frac{x+d}{d}$$

$$\boldsymbol{E} = -\frac{dV}{dx}\boldsymbol{a}_x = -\frac{V_o}{(x+d)\ln 2}\boldsymbol{a}_x$$

(b) $$\boldsymbol{P} = (\varepsilon_r - 1)\varepsilon_o \boldsymbol{E} = -\left(\frac{x+d}{d} - 1\right)\frac{\varepsilon_o V_o}{(x+d)\ln 2}\boldsymbol{a}_x = -\frac{\varepsilon_o x V_o}{d(x+d)\ln 2}\boldsymbol{a}_x$$

(c)



$$\rho_{ps}\,|_{x=0} = \boldsymbol{P} \bullet (-\boldsymbol{a}_x)\,|_{x=0} = 0$$

$$\rho_{ps}\,|_{x=d} = \boldsymbol{P} \bullet \boldsymbol{a}_x\,|_{x=d} = -\frac{\varepsilon_o V_o}{2d\ln 2}$$

(d) $$\boldsymbol{E} = \frac{\rho_s}{\varepsilon}\boldsymbol{a}_x = \frac{Q}{\varepsilon S}\boldsymbol{a}_x = \frac{Q}{\varepsilon_o(1+\frac{x}{d})S}\boldsymbol{a}_x$$

$$V = -\int \boldsymbol{E} \bullet d\boldsymbol{l} = -\frac{Q}{\varepsilon_o S}\int_a^d \frac{dx}{(1+\frac{x}{d})} = \frac{Q}{\varepsilon_o S} d\ln 2$$

$$C = \frac{Q}{V} = \frac{\varepsilon_o S}{d\ln 2}$$

Prob. 6.56

We solve Laplace's equation for an inhomogeneous medium.

$$\nabla \bullet (\varepsilon \nabla V) = \frac{d}{dx}\left(\varepsilon \frac{dV}{dx}\right) = 0 \qquad \longrightarrow \qquad \varepsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\varepsilon} = \frac{A}{2\varepsilon_o}\left[1+\left(\frac{x}{d}\right)^2\right]$$

$$V = \frac{A}{2\varepsilon_o}(x + \frac{x^3}{3d^2}) + B$$

When x=d, V=V_o,

$$V_o = \frac{A}{2\varepsilon_o}(d + \frac{d}{3}) + B \qquad \longrightarrow \qquad V_o = \frac{2Ad}{3\varepsilon_o} + B \qquad (1)$$

When x = -d, V=0,

$$0 = \frac{A}{2\varepsilon_o}(-d - \frac{d}{3}) + B \qquad \longrightarrow \qquad 0 = -\frac{2Ad}{3\varepsilon_o} + B \qquad (2)$$

Adding (1) and (2), $\quad V_o = 2B \quad \longrightarrow \quad B = V_o / 2$

From (2),

$$B = \frac{2Ad}{3\varepsilon_o} = \frac{V_o}{2} \qquad \longrightarrow \qquad A = \frac{3\varepsilon_o V_o}{4d}$$

$$E = -\nabla V = -\frac{dV}{dx}\mathbf{a}_x = -\frac{A}{\varepsilon}\mathbf{a}_x = -\frac{3\varepsilon_o V_o}{4d}\frac{\left[1+\left(\frac{x}{d}\right)^2\right]}{2\varepsilon_o}\mathbf{a}_x = -\frac{-3V_o}{8d}\left[1+\left(\frac{x}{d}\right)^2\right]\mathbf{a}_x$$

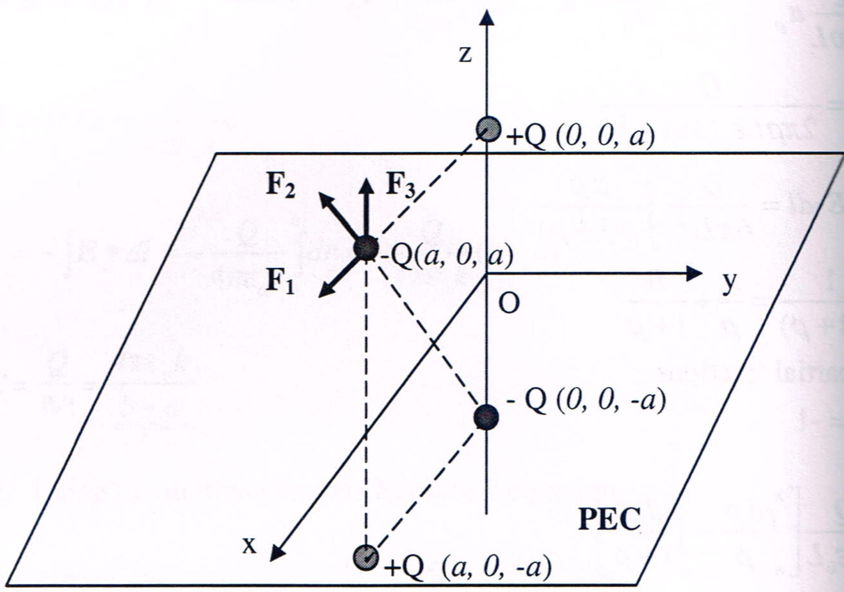
$$\rho_s = D \bullet a_n = \varepsilon E \bullet a_x \Big|_{x=d} = -A = -\frac{3\varepsilon_o V_o}{4d}$$

$$Q = \int_S \rho_s dS = \rho_s S = -\frac{3S\varepsilon_o V_o}{4d}$$

$$C = \frac{|Q|}{V_o} = \underline{\underline{\frac{3\varepsilon_o S}{4d}}}$$

Prob. 6.60

The images are shown with proper sign at proper locations. Figure does not show the actual direction of forces but they are expressed a follows:



$$\mathbf{F}_1 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{-\mathbf{a}_x}{a^2}\right]$$

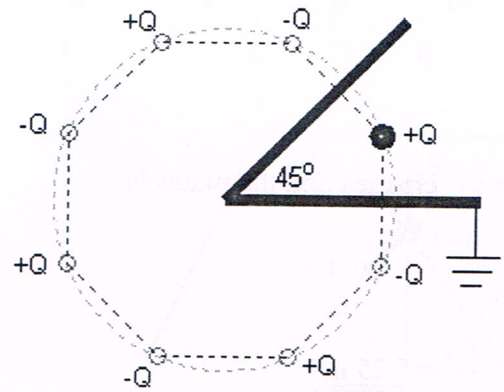
$$\mathbf{F}_2 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{a\mathbf{a}_x + 2a\mathbf{a}_z}{\left(\sqrt{a^2+4a^2}\right)^3}\right]$$

$$\mathbf{F}_3 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{-\mathbf{a}_z}{4a^2}\right]$$

$$\mathbf{F}_{\text{total}} = \frac{Q^2}{4\pi\varepsilon_o a^2}\left[\left(\frac{1}{5\sqrt{5}}-1\right)\mathbf{a}_x + \left(\frac{2}{5\sqrt{5}}-\frac{1}{4}\right)\mathbf{a}_z\right]$$

$$= \frac{Q^2}{4\pi\varepsilon_o a^2}\left[-0.91\boldsymbol{a}_x - 0.071\boldsymbol{a}_y\right] \text{ N}$$

Prob. 6.63



$$N = \left(\frac{360^o}{45^o} - 1 \right) = \underline{\underline{7}}$$

Prob. 6.64

(a)

$$\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- = \frac{\rho_L}{2\pi\varepsilon_o}\left(\frac{\boldsymbol{a}_{\rho 1}}{\rho_1} - \frac{\boldsymbol{a}_{\rho 2}}{\rho_2}\right) = \frac{16\times10^{-9}}{2\pi\times\dfrac{10^{-9}}{36\pi}}\left[\frac{(2,-2,3)-(3,-2,4)}{|(2,-2,3)-(3,-2,4)|^2} - \frac{(2,-2,3)-(3,-2,-4)}{|(2,-2,3)-(3,-2,-4)|^2}\right]$$

$$= 18x16\left[\frac{(-1,0,-1)}{2} - \frac{(-1,0,7)}{50}\right] = \underline{\underline{-138.2\boldsymbol{a}_x - 184.3\boldsymbol{a}_y \text{ V/m}}}$$

(b) $\rho_s = D_n$

$$\boldsymbol{D} = \boldsymbol{D}_+ + \boldsymbol{D}_- = \frac{\rho_L}{2\pi}\left(\frac{\boldsymbol{a}_{\rho 1}}{\rho_1} - \frac{\boldsymbol{a}_{\rho 2}}{\rho_2}\right) = \frac{16x10^{-9}}{2\pi}\left[\frac{(5,-6,0)-(3,-6,4)}{|(5,-6,0)-(3,-6,4)|^2} - \frac{(5,-6,0)-(3,-6,-4)}{|(5,-6,0)-(3,-6,-4)|^2}\right]$$

$$= \frac{8}{\pi}\left[\frac{(2,0,-4)}{20} - \frac{(2,0,4)}{20}\right] \text{ nC/m}^2 = -1.018\boldsymbol{a}_z \text{ nC/m}^2$$

$$\underline{\underline{\rho_s = -1.018 \text{ nC/m}^2}}$$