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Linear superposition principle for partially coherent solitons

Sergey A. Ponomarenko
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

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The existence of a linear superposition principle is demonstrated for partially coherent solitons with identical intensity profiles that are supported by the same medium. Since such degenerate partially coherent solitons are generic for saturable as well as for Kerr-like nonlinear media, our results are relevant to any noninstantaneous nonlinear media. The proposed superposition principle suggests a physical interpretation of partially coherent solitons as generalized linear modes of their self-induced waveguides. The power of such a superposition principle is illustrated by identifying soliton structures with controllable coherence properties both in logarithmically saturable and in Kerr-like nonlinear media.

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In recent years, partially coherent spatial solitons have attracted much attention, especially after such solitons had been successfully generated in pioneering experiments using biased photorefractive crystals [1]. To date, three theoretical approaches have been developed to study partially coherent solitons. The first approach relies on the nonlinear propagation equation for the equal-time correlation function of a partially coherent beam [2–4]. The second method introduces the so-called coherent density function [5,6], whereas the third is the self-consistent multimode approach [7–9]. Recently, all the three methods have been shown to be equivalent [10].

The self-consistent multimode method has been especially fruitful because not only is it best suited for identifying families of stationary solitons existing in saturable [11–13] as well as in Kerr-like media [14–20], but it also provides a valuable physical insight into the structure of partially coherent solitons. In essence, this approach views a partially coherent soliton as an incoherent superposition of mutually uncorrelated, nonlinear modes of the waveguide which is induced in the medium via nonlinearity. This view imparts physical intuition by making it possible to draw analogies with the theory of linear waveguides [21]. In particular, in the case of a linear multimode waveguide, any beam launched into the waveguide can be represented as a linear superposition of the modes of the waveguide. In this connection, it should be noted that in many cases, the same self-induced waveguide can be shown to support more than one partially coherent soliton with the same intensity profile [13,16,18]. By analogy with the linear waveguide theory, one may then ask whether a *linear* superposition of such degenerate partially coherent solitons can under certain circumstances result in another soliton trapped by the same waveguide.

In this Rapid Communication, we show that any linear superposition of degenerate partially coherent solitons, which preserves the total intensity of the self-induced waveguide, results in yet another soliton supported by the waveguide. The existence of such a superposition principle provides a new insight into the physical nature of partially coherent solitons. On the one hand, such solitons can be treated as multisoliton complexes that reshape upon collisions [15,19]. On the other hand, each degenerate partially coherent soliton can be viewed as a generalized linear mode of the self-induced waveguide, and it serves as an elementary building block for more complicated soliton structures. It should be emphasized that the present approach differs from previous studies of multisoliton complexes [15,19] in two respects. First of all, while explicit analytic results of Refs. [15,19] apply to integrable, Kerr-like systems, our superposition principle holds for any nonlinear medium supporting degenerate partially coherent solitons. Second, the authors of [19] consider multisoliton complexes with the fundamental solitons (or the nonlinear modes) that can be mutually either correlated or uncorrelated. In our case, however, all of the nonlinear modes of the self-induced waveguide are mutually uncorrelated, but there are correlations among the degenerate partially coherent solitons which are linearly superposed.

We also demonstrate that linear superpositions of partially coherent solitons in saturable and in nonsaturable media result in a variety of soliton structures with modified coherence properties. The particular examples of solitons in logarithmically saturable and in Kerr-like media are considered. Further, the *coherent* nature of such a linear superposition leads to *interference* effects among degenerate partially coherent solitons, which are manifested in the change of the effective soliton spatial coherence length. The spatial coherence length is a fundamental characteristic of a partially coherent soliton, which has been shown to define the threshold for modulational [22] and transverse [23] instabilities as well as that of a collapse of such a soliton in inertial Kerr media [24]. Therefore, the proposed superposition principle may provide a useful tool for producing partially coherent solitons with desirable properties.

To begin, we consider a paraxial partially, coherent beam propagating in a noninstantaneous nonlinear medium along the z axis. The second-order statistical properties of such a beam are fully specified by the equal-time correlation function $\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z)$ (Chap. 4 of [25]) at a pair of points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ in the plane transverse to the z axis. The propagation of the equal-time correlation function in the nonlinear regime is governed by the generalized Wolf's equation [2,10]

$$2ik\partial_z\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z)+(\nabla_{\perp 1}^2-\nabla_{\perp 2}^2)\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z)+k^2[n_{nl}^2(I_1)-n_{nl}^2(I_2)]\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z)=0, \tag{1}$$

where $n_{nl}(I)$ is the nonlinear refractive index, $I_j \equiv I(\boldsymbol{\rho}_j)$, $j=1,2$, and $\boldsymbol{\nabla}_\perp$ is a gradient transverse to the direction of propagation of the beam.

In order for a partially coherent beam to be a soliton, it ought to induce the waveguide whose modes can reproduce its intensity profile. Suppose now that the self-induced waveguide can support more than one soliton with the same intensity profile. We label such a degeneracy by the set of indices $\{\nu\}$. It follows from Eq. (1) that the equal-time correlation function $\Gamma_\nu(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ of each degenerate soliton obeys the equation

$$(\nabla_{\perp 1}^2-\nabla_{\perp 2}^2)\Gamma_\nu(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=k^2[n_{nl}^2(I_2)-n_{nl}^2(I_1)]\Gamma_\nu(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2), \tag{2}$$

which conveys the simple physical idea of balance between the nonlinear refraction and diffraction. It should be kept in mind that in Eq. (2)

$$I_j \equiv I(\boldsymbol{\rho}_j)=\Gamma_\nu(\boldsymbol{\rho}_j,\boldsymbol{\rho}_j), \tag{3}$$

where $I(\boldsymbol{\rho})$ is the common intensity profile.

Consider next the linear superposition of equal-time correlation functions of the form $\Gamma_s(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\Sigma_\nu c_\nu \Gamma_\nu(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$, where $\{c_\nu\}$ are constant coefficients. One can readily infer from Eqs. (2) and (3) that Γ_s will satisfy the same equation (2) as every Γ_ν does, provided the intensity of the resulting soliton at every point $\boldsymbol{\rho}$ is equal to the intensity of each component, i.e.,

$$I_s(\boldsymbol{\rho})=I(\boldsymbol{\rho}). \tag{4}$$

Condition (4) is equivalent to the constraint on the values of the set of coefficients $\{c_\nu\}$, $\Sigma_\nu c_\nu=1$. Thus we have proven the following assertion.

Theorem 1. Any linear superposition of partially coherent solitons with the equal-time correlation functions $\{\Gamma_\nu(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)\}$ which have the same intensity profile $I(\boldsymbol{\rho})$, is also a soliton with the equal-time correlation function given by the expression

$$\Gamma_s(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\sum_\nu c_\nu \Gamma_\nu(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2), \tag{5}$$

provided that the values of the coefficients $\{c_\nu\}$ satisfy the condition

$$\sum_\nu c_\nu=1. \tag{6}$$

To appreciate the physical significance of Eq. (6), recall that in general, the effective width of the soliton intensity profile depends on the soliton power, $P=\int d^2\rho I(\boldsymbol{\rho})$. Therefore, condition (6) implies that the total power must be the same in order to maintain the same effective soliton width. Another point to bear in mind is that the coefficients c_ν's need not be numerical constants. In fact they can be expressed in terms of $N-1$ free parameters in the case of N-fold degeneracy of the self-induced waveguide. In particular, one can choose $N-1$ angles, which fix the position of a unit vector $\mathbf{n}$ in the N-dimensional parametric space. The constraint (6) can then be rewritten as $\Sigma_\nu n_\nu^2=1$. It follows from this geometric argument that all possible compound solitons correspond to the points on the surface of a unit sphere in such a parametric space.

We now turn to the illustrations of the proposed superposition principle. Consider first the logarithmically saturable medium with the nonlinear refractive index n_{nl} of the form

$$n_{nl}^2(I)=(\Delta n)^2\ln(1+I/I_t), \tag{7}$$

where Δn and I_t are constants specified by the material. It was shown in Ref. [13] that in such a medium, the equal-time correlation function of the most general partially coherent soilton with a circularly symmetric intensity profile, the twisted Gaussian Schell-model soliton (TGSM), has the form

$$\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=I_0\exp\left(-\frac{\rho_1^2+\rho_2^2}{4\sigma_I^2}\right)\exp\left[-\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2}{2\sigma_c^2}\right] \times\exp(iu|\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2|). \tag{8}$$

Here I_0 is the peak intensity of the soliton, and the soliton width σ_I, the soliton coherence length σ_c, and the twist parameter u are related by the self-consistency condition

$$\frac{\sigma_I}{\sigma_c}=\left[\frac{(\alpha^2-1)/2}{1+\sqrt{1+u^2\sigma_c^4(\alpha^2-1)}}\right]^{1/2}, \tag{9}$$

where $\alpha^2=2k^2\sigma_I^2(\Delta n)^2$. Moreover, the twist parameter satisfies the inequality $-1/\sigma_c^2\leqslant u\leqslant 1/\sigma_c^2$ [13].

Let us introduce the equal-time degree of spatial coherence of a soliton at a pair of points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ defined by the expression (Ref. [25] [Sec.4.2])

$$\gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\frac{\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)}{\sqrt{I(\boldsymbol{\rho}_1)}\sqrt{I(\boldsymbol{\rho}_2)}}. \tag{10}$$

The analysis of Eq. (9) indicates that (i) given the values of σ_I and σ_c, there exists twofold degeneracy of TGSM solitons with respect to the sign of u, (ii) for a fixed soliton width σ_I, there exists a continuum of the pairs of (σ_c,u) that satisfy Eq. (9). We will employ only the first type of degeneracy. In particular, let us consider a superposition with $c_1=c_2=1/2$ of TGSM solitons with $\sigma_{c1}=\sigma_{c2}\equiv\sigma_c$ and $u_1=-u_2\equiv u$. It follows at once from Eqs. (5), (8), and (10) that the equal-time degree of spatial coherence of the resulting soliton is given by

$$\gamma_s(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\exp\left[-\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2}{2\sigma_c^2}\right]\cos(u|\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2|). \tag{11}$$

Further, we define the effective coherence length of a $(2+1)D$ soliton by the expression

$$l_c^2(\boldsymbol{\rho})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\delta_x d\delta_y}{\pi}|\gamma_s(x,y;x+\delta_x,y+\delta_y)|^2. \tag{12}$$

Substituting from Eq. (11) into Eq. (12), one obtains for the coherence length of the equally weighed superposition of TGSM solitons with $\sigma_{c1}=\sigma_{c2}\equiv\sigma_c$ and $u_1=-u_2\equiv u$ the expression

$$l_c(\boldsymbol{\rho})=\sigma_c\left(\frac{1+e^{-u^2\sigma_c^2\rho^2}}{2}\right)^{1/2}. \tag{13}$$

Two conclusions can be drawn from Eq. (13). First of all, any superposition of TGSM solitons results in the dependence of the coherence length at a particular point on the location of the point within the self-trapped beam. This is a novel feature for solitons in the logarithmic medium. Second, the coherence length of the resulting soliton is never greater than the coherence length of a TGSM soliton, $l_c(\rho)\leqslant\sigma_c$, and l_c can be controlled by varying the twist parameter.

The existence of degenerate partially coherent solitons was also demonstrated in both defocusing [16] and focusing [18] Kerr-like media. In either case, stable partially coherent solitons have been shown to consist of bound modes as well as of the continuum of radiation modes whose distribution depends on the angular spectrum of a partially coherent source. In this paper, we consider only $(1+1)D$ dark, degenerate solitons with the intensity profile of the form

$$I(\eta)=I_0[1-\epsilon^2\,\mathrm{sech}^2(\eta)]. \tag{14}$$

Here I_0 is the background intensity, $\epsilon^2\leqslant 1$ specifies the degree of grayness of the soliton, and $\eta=x/x_0$, where x_0 is the soliton width. It was shown in Ref. [16] that if $x_0=2/k_0\epsilon^2 n_2 I_0$, there is only one bound mode given by the expression

$$U_b(\eta)=\mathrm{sech}(\eta), \tag{15a}$$

and the allowed even and odd radiation modes can be represented as

$$U_e(\eta,Q)=Q\cos(Q\eta)-\sin(Q\eta)\tanh(\eta), \tag{15b}$$

$$U_o(\eta,Q)=Q\sin(Q\eta)+\cos(Q\eta)\tanh(\eta). \tag{15c}$$

Here $k_0=k/n_0$ is the free-space wave vector, n_0 and n_2 are the linear part of the refractive index and the nonlinear Kerr coefficient, respectively. Also, $Q^2=[k_0^2(n_0^2-n_2I_0)-\beta^2]x_0^2>0$, where β is the mode propagation constant.

An ensemble realization of the electric field $E(\eta,z)$ can then be written as

$$E(\eta,z)=c_bU_b(\eta)e^{i\beta_b z}+\int dQ[c_e(Q)U_e(\eta,Q)+c_o(Q)U_o(\eta,Q)]e^{i\beta_r(Q)z}, \tag{16}$$

where c_b and $c_{e,o}(Q)$ are the random temporal coefficients of the bound and radiation modes, and β_b as well as $\beta_r(Q)$ are the corresponding propagation constants. To achieve self-trapping, one has to impose the following statistics of the temporal coefficients:

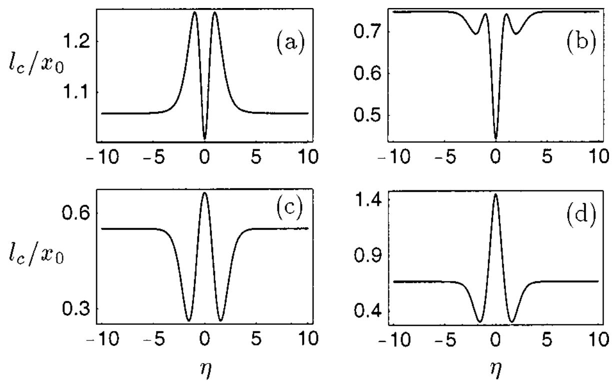

FIG. 1. The coherence length of the superposition of two dark solitons as a function of the position η within the soliton for different values of the mixing angle θ: (a) $\theta=0$, (b) $\theta=\pi/6$, (c) $\theta=\pi/3$, and (d) $\theta=\pi/2$. The numerical values of the parameters are $\epsilon^2=0.5$, $\sigma_1=\sigma_2=Q_0=0.7$.

$$\langle|c_b|^2\rangle=\lambda_b, \tag{17a}$$

$$\langle c_e^*(Q)c_o(Q')\rangle=\langle c_b^*(Q)c_{e,o}(Q')\rangle=0, \tag{17b}$$

$$\langle c_e^*(Q)c_e(Q')\rangle=\langle c_o^*(Q)c_o(Q')\rangle=\mathcal{D}(Q)\delta(Q-Q'), \tag{17c}$$

where λ_b is the weight of the bound mode, and $\mathcal{D}(Q)$ is the distribution function of radiation modes. One can show that for any given $\mathcal{D}(Q)$, the modes reproduce the intensity profile of Eq. (14) provided that $I_0=\int dQ(1+Q^2)\mathcal{D}(Q)$, and $\lambda_b=\int dQ\mathcal{D}(Q)-\epsilon^2 I_0$.

Let us consider two possible distributions of radiation modes:

$$\mathcal{D}_1(Q)=Z_1e^{-Q/\sigma_1},\quad 0\leqslant Q\leqslant Q_{max} \tag{18a}$$

$$\mathcal{D}_2(Q)=Z_2e^{-(Q-Q_0)^2/2\sigma_2^2},\quad |Q|\leqslant Q_{max}. \tag{18b}$$

where $\sigma_{1,2}$ is the width of the corresponding distribution, $Q_{max}=k_0x_0\sqrt{n_0^2-n_2I_0}$. A particular distribution $\mathcal{D}(Q)$ is specified by the angular spectrum of the light source. Thus in order to generate, for example, the distribution defined in expression (18a), one has to prepare a partially coherent source whose angular spectrum decreases with the launch angle [26], whereas to produce the distribution of Eq. (18b), one should use a light source with the angular spectrum peaked around a finite angle corresponding to the spatial frequency Q_0. Consider now a linear superposition with $c_1(\theta)=\cos^2\theta$ and $c_2(\theta)=\sin^2\theta$ of degenerate solitons corresponding to $\mathcal{D}_1(Q)$ and $\mathcal{D}_2(Q)$. The equal-time degree of spatial coherence of the compound soliton at the points with coordinates η and $\eta+\delta$ can be represented as

$$\gamma_{tot}(\eta,\eta+\delta)=\sum_{j=1,2}c_j(\theta)\gamma_j(\eta,\eta+\delta). \tag{19}$$

Here θ is a free parameter, the "mixing angle," and $\gamma_j(\eta,\eta+\delta)$ is the equal-time degree of spatial coherence of each degenerate soliton. One can then determine the soliton coherence length defined by the expression

$$l_{tot}(\eta)=\int_{-\infty}^{\infty}d\delta|\gamma_{tot}(\eta,\eta+\delta)|^2. \qquad (20)$$

In Fig. 1, we have displayed the coherence length of the compound soliton as a function of the position η for different values of the mixing angle θ. The limiting cases $\theta=0$ and $\theta=\pi/2$ correspond to the presence of only one partially coherent soliton in superposition (19). It is seen from the figure that on varying the mixing angle, one can significantly modify the behavior of the coherence length. In particular, due to the interference of the soliton components, the coherence length of the compound soliton can reach either minimum or maximum at $\eta=0$ depending on whether the contribution of the odd radiation modes or those of the even radiation modes and the bound mode dominate in the vicinity of this point.

In summary, we have discovered a linear superposition principle for partially coherent solitons with the same intensity profile. Our arguments hold for any such degenerate solitons regardless of the functional form of the nonlinearity of the medium. The present approach offers an alternative view of partially coherent solitons as generalized linear modes of the self-induced waveguide. The interference of such generalized modes may be useful in designing spatial solitons with prescribed coherence properties. Numerical examples are considered for the solitons in saturable logarithmic as well as in Kerr-like nonlinear media.

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