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Twist phase and classical entanglement of partially coherent light

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We demonstrate that the presence of a twist phase in a random light beam leads to classical entanglement between phase space degrees of freedom of the beam. We find analytically the bi-orthogonal decomposition of the Wigner function of a twisted Gaussian Schell-model (TGSM) source and quantify its entanglement by evaluating the Schmidt number of the decomposition. We show that (i) classical entanglement of a TGSM source vanishes concurrently with the twist in the fully coherent limit and (ii) entanglement dramatically increases as the source coherence level decreases. We also show that the discovered type of classical entanglement of a Gaussian Wigner function does not degrade on beam propagation in free space.

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Classical entanglement has recently attracted much attention both on the fundamental side [1–3] and in light of its promise for a multitude of applications from quantum communication protocols [4–6], optical metrology [7–10], and mimicking quantum processes with classical light [11] to optical communications through noisy environments, such as a biological tissue [12] or the turbulent atmosphere [13]. The theoretical foundation of classical entanglement and its place in the context of quantum information processing was laid down by Spreeuw in a series of seminal papers [1,14]. This pioneering theoretical work was then followed by the experimental demonstration and exploration of entanglement between the spatial and polarization degrees of freedom (DoFs) of a radially or azimuthally polarized classical light beam by several groups [4,15–17]; see also [3] for an up-to-date review.

Although the vast majority of research on classical entanglement to date has been concerned with spatial and polarization DoFs of non-uniformly polarized vector beams, other avenues have also been explored, including space-frequency entanglement of optical fields in connection with either atom trapping [18] or phase-space quality of ultrashort pulsed beams [19] and space–time entanglement of classical light [20]. Further, an instructive perspective on classical entanglement in the phase space of a fully coherent beam endowed with a vortex phase has been theoretically presented in [21] and experimentally verified in [22].

At the same time, the cross-spectral density of a random beam field can possess an entirely different, non-local phase, a twist phase, which couples pairs of points within the transverse plane of the beam. The twist phase, which is unique to partially coherent light as it vanishes in the fully coherent limit, was introduced theoretically in [23] in the context of twisted Gaussian Schell-model (TGSM) beams. Such beams were first experimentally realized in [24], followed by the advancement of an alternative experimental protocol to generate genuine TGSM beams [25]. Partially coherent light fields endowed with a twist phase have been shown to arise in light–matter interactions in certain nonlinear media [26,27], and they have been successfully employed for information transfer through the turbulent atmosphere [28,29] and optical imaging [30,31].

In this Letter, we demonstrate that endowing the optical field of a statistical beam with a twist phase leads to a new kind of classical entanglement in the phase space of the beam. We quantify the discovered entanglement by evaluating the Schmidt number of a bi-orthogonal decomposition of the Wigner function of a TGSM beam at the source. In particular, we show that the twist engendered entanglement disappears with the twist in the fully coherent limit. Concurrently, we demonstrate that the strength of entanglement of a TGSM source is significantly enhanced as the source coherence falls off. We also show that the discovered entanglement of a TGSM source does not degrade as the beam, generated by such a source, propagates in free space.

We start by recalling that the cross-spectral density of a TGSM source at a pair of points $\mathbf{r}_1$ and $\mathbf{r}_2$ in the source plane is given by [23]

$$W(\mathbf{r}_1, \mathbf{r}_2) \propto \exp\left(-\frac{\mathbf{r}_1^2+\mathbf{r}_2^2}{4\sigma_I^2}\right) \exp\left[-\frac{(\mathbf{r}_1-\mathbf{r}_2)^2}{2\sigma_c^2}\right] e^{iu(\mathbf{r}_1\times\mathbf{r}_2)_\perp}, \tag{1}$$

where $(\mathbf{r}_1 \times \mathbf{r}_2)_\perp = x_1 y_2 - x_2 y_1$ is a cross product of 2D vectors in the transverse plane of the source. Further, we introduced an rms width σ_I of the source intensity, source transverse coherence width σ_c, and a twist parameter u. Hereafter, we drop any immaterial normalization factor. It will prove advantageous to work with the Wigner function in lieu of the cross-spectral density. The former is defined by the expression

$$\mathcal{W}(\mathbf{R}, \mathbf{k}) = \int d\mathbf{r}\, W(\mathbf{R}-\mathbf{r}/2, \mathbf{R}+\mathbf{r}/2) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2}$$

where

$$\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2, \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \tag{3}$$

are the radius vectors of the "center-of-mass" and position difference of any two points with coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ within the source. On substituting from Eq. (1) into Eq. (2) and performing elementary Gaussian integrations, we can show that

$$\mathcal{W}(\mathbf{R}, \mathbf{k}) = \mathcal{W}_+(X, k_y)\mathcal{W}_-(Y, k_x). \tag{4}$$

Here,

$$\mathcal{W}_+(X, k_y) \propto e^{-X^2/2\sigma_I^2} e^{-(k_y+uX)^2\sigma_{\text{eff}}^2/2}, \tag{5}$$

as well as

$$\mathcal{W}_-(Y, k_x) \propto e^{-Y^2/2\sigma_I^2} e^{-(k_x-uY)^2\sigma_{\text{eff}}^2/2}, \tag{6}$$

and we introduced the notation

$$\frac{1}{\sigma_{\text{eff}}^2} = \frac{1}{\sigma_c^2} + \frac{1}{4\sigma_I^2}. \tag{7}$$

Next, we infer from Eqs. (5) and (6) that the twist phase engenders entanglement between k_y and X and, independently, between k_x and Y because $\mathcal{W}_+$ cannot be separated into a product of a function of the spatial coordinate X and that of the wave vector component k_y as long as $u \neq 0$. By the same token, $\mathcal{W}_-$ is entangled as a function of k_x and Y. We can then streamline the notation by introducing

$$x_+ = X, \quad k_+ = k_y; \tag{8}$$

$$x_- = Y, \quad k_- = k_x \tag{9}$$

and writing down the two Wigner functions in a common form as

$$\mathcal{W}_\pm(x_\pm, k_\pm) \propto e^{-x_\pm^2/2\sigma_I^2} e^{-(k_\pm \pm u x_\pm)^2\sigma_{\text{eff}}^2/2}. \tag{10}$$

The Wigner function of Eq. (10) can be expanded into a bi-orthogonal series [32]

$$\mathcal{W}_\pm(x_\pm, k_\pm) = \sum_n \lambda_{n\pm}\psi_n(x_\pm)\phi_n(k_\pm), \tag{11}$$

where the two orthogonal sets of real eigenfunctions $\{\psi_n\}$ and $\{\phi_n\}$ form a bi-orthogonal set in the phase space. Note that as the Wigner function is real, so are the eigenfunctions $\{\psi_n\}$ and $\{\phi_n\}$. At the same time, $\{\lambda_n\}$ need not be all positive because the Wigner function is not necessarily positive definite [33].

In general, the two sets $\{\psi_n\}$ and $\{\phi_n\}$ must be determined by solving a set of coupled Fredholm integral equations [32], which is a formidable task. Fortunately, for Gaussian Wigner functions, we can circumnavigate this obstacle by introducing a scaling transformation

$$\tilde{x}_\pm = a x_\pm, \quad \tilde{k}_\pm = b k_\pm. \tag{12}$$

It follows that provided

$$a = \frac{1}{\sqrt{2}\sigma_I}, \quad b = \frac{\sigma_c}{\sqrt{2(1 + t^2/\xi_c^2 + \xi_c^2/4)}}, \tag{13}$$

as well as

$$\zeta_\pm = \mp\frac{t/\xi_c}{\sqrt{1 + t^2/\xi_c^2 + \xi_c^2/4}}, \tag{14}$$

we can cast the Wigner function of Eq. (10) into a "canonical" form

$$\mathcal{W}_\pm(\tilde{x}_\pm, \tilde{k}_\pm) \propto \exp\left(\frac{2\tilde{x}_\pm\tilde{k}_\pm\zeta_\pm - \tilde{x}_\pm^2 - \tilde{k}_\pm^2}{1 - \zeta_\pm^2}\right). \tag{15}$$

Here, we introduced dimensionless twist t and coherence ξ_c parameters by the expressions

$$t = u\sigma_c^2, \quad \xi_c = \sigma_c/\sigma_I. \tag{16}$$

We note in passing that these two propagation invariant parameters completely characterize any TGSM source. Furthermore, t is known to be bounded, $|t| \leq 1$, such that u vanishes in the fully coherent limit, $\sigma_c \to \infty$ [23].

To proceed, we can apply the Mehler formula to the "canonical" Gaussian Wigner function [32]

$$\exp\left(\frac{2\tilde{x}_\pm\tilde{k}_\pm\zeta_\pm - \tilde{x}_\pm^2 - \tilde{k}_\pm^2}{1 - \zeta_\pm^2}\right) = \sqrt{1 - \zeta_\pm^2}\, e^{-\tilde{x}_\pm^2 - \tilde{k}_\pm^2} \times \sum_{n=0}^{\infty} \frac{\zeta_\pm^n}{2^n n!} H_n(\tilde{x}_\pm) H_n(\tilde{k}_\pm), \tag{17}$$

where $H_n(x)$ is a Hermite polynomial of the order n, to read off the bi-orthogonal set $\{\psi_n\}$ and $\{\phi_n\}$ by examining Eqs. (11), (15), and (17). We can infer from these equations that up to an irrelevant normalization constant

$$\psi_n(\tilde{x}_\pm) \propto \left(\frac{1}{2^n n!}\right)^{1/2} H_n(\tilde{x}_\pm) e^{-\tilde{x}_\pm^2/2}; \tag{18}$$

$$\phi_n(\tilde{k}_\pm) \propto \left(\frac{1}{2^n n!}\right)^{1/2} H_n(\tilde{k}_\pm) e^{-\tilde{k}_\pm^2/2}, \tag{19}$$

and the eigenvalues $\{\lambda_n\}$ are given by

$$\lambda_{n\pm} = \sqrt{1 - \zeta_\pm^2}\,\zeta_\pm^n. \tag{20}$$

Next, we can choose a non-negative eigenvalue set to quantify classical entanglement [3]. Specifically, taking

$$\nu_n = \lambda_{n\pm}^2 = (1 - \nu)\nu^n, \tag{21}$$

where we denoted

$$\nu = \frac{t^2/\xi_c^2}{1 + t^2/\xi_c^2 + \xi_c^2/4}, \tag{22}$$

we can define a Schmidt number K of the decomposition of Eq. (11) as

$$K = \frac{\left(\sum_{n=0}^{\infty} \nu_n\right)^2}{\sum_{n=0}^{\infty} \nu_n^2}. \tag{23}$$

Further, on substituting from Eqs. (21) and (22) into Eq. (23), we obtain, after elementary algebra, the expression

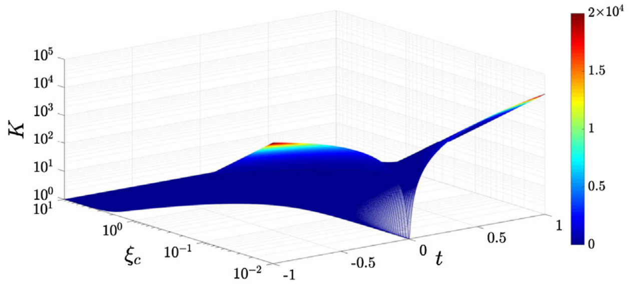

Fig. 1. Schmidt number K as function of the coherence parameter ξ_c in the log-log scale and of the twist parameter t. Notice that the figure illustrates two complementary trends: very rapid growth of K in the vicinity of $\xi_c = 0$ and slow decay of K in the remainder of the available parameter range of ξ_c. The largest values of K are concentrated at the edges corresponding to the largest attainable twist $t = \pm 1$ and the lowest available values of ξ_c.

$$K = \frac{1 + 2t^2/\xi_c^2 + \xi_c^2/4}{1 + \xi_c^2/4}. \quad \textbf{(24)}$$

Equation (24) is the central result of this Letter. We can conclude at once from Eq. (24) that entanglement vanishes with the twist: $t = 0$ implies $K = 1$. At the same time, for any finite twist strength, $t \neq 0$, the Schmidt number approaches unity, as ξ_c tends to infinity, i.e., a highly coherent TGSM source manifests very weak entanglement. On the other hand, as $\xi_c \to 0$, $K \simeq 2t^2/\xi_c^2 \to \infty$. That is, a nearly incoherent TGSM source is highly classically entangled.

To address a laboratory implementation of the twist induced classical entanglement, we note that it is difficult, although not impossible [34], to design an optical source with a transverse coherence length shorter than the wavelength of light under usual laboratory conditions. Moreover, a garden variety partially coherent source is produced by transmitting a fully coherent beam through a rotating ground glass disc with random imperfections, whereby the transverse coherence size of thereby generated source is determined by a characteristic distance between adjacent imperfections. The latter is typically bounded from below by the disc design to about 0.1 mm [25]. Thus, assuming that $\sigma_c \sim 0.1$ mm and considering a well expanded light beam with $\sigma_I \sim 1$ cm, we can estimate a readily attainable level of twist phase induced classical entanglement (Schmidt number) as $K_{\max} \simeq 2\sigma_I^2/\sigma_c^2 \sim 2 \times 10^4$. This fairly typical situation is illustrated in Fig. 1, where we exhibit the behavior of K as a function of the coherence parameter ξ_c in the log-log scale to better juxtapose the rapid growth of the Schmidt number in the low-coherence limit to the asymptotic fall off of K toward unity. In the same figure, we also display the dependence of the Schmidt number on the twist parameter t.

Finally, we recall that t and ξ_c are invariant on paraxial propagation of a TGSM beam generated by the source: u, σ_c, and σ_I scale on propagation such that t and ξ_c remain constant [23]. Hence, the Schmidt number and, by implication, the degree of classical entanglement of the beam are propagation constants. In other words, the strength of the unveiled, twist phase induced type of classical entanglement of a Gaussian Wigner function remains invariant on beam evolution in free space.

In conclusion, we have discovered and quantified classical entanglement induced in the phase space of a random beam by endowing the field of the beam with a twist phase. Although our explicit results are for TGSM beams, the twist phase can be shown to be imposed on any statistical beam [35–38]. The twist induced entanglement then is relevant to random beams described by any Wigner function, even though, in general, the strength of such entanglement will not remain invariant on beam propagation in free space. The disappearance of the discovered entanglement in the fully coherent limit sets it apart from any other known type of classical or quantum entanglement that exists for coherent and partially coherent beams or for pure and mixed quantum states, respectively. Our results can find applications to quantum information processing, metrology, and optical communications with classically entangled light.

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Data Availability. Data underlying the results presented in this Letter are not publicly available at this time but may be obtained from the authors upon reasonable request.

REFERENCES

1. R. J. C. Spreeuw, Found. Phys. **28**, 361 (1998).
2. X.-F. Qian, A. N. Vamivakas, and J. H. Eberly, Opt. Photon. News **28**, 34 (2017).
3. A. Forbes, A. Aiello, and B. Ndagano, Prog. Opt. **64**, 99 (2019).
4. E. Karimi, J. Leach, S. Slussarenko, B. Piccirillo, L. Marrucci, L. Chen, W. She, S. Franke-Arnold, M. J. Padgett, and E. Santamato, Phys. Rev. A **82**, 022115 (2010).
5. I. Nape, B. Ndagano, and A. Forbes, Phys. Rev. A **95**, 053859 (2017).
6. B. Ndagano, I. Nape, M. A. Cox, C. Rosales-Guzman, and A. Forbes, J. Lightwave Technol. **36**, 292 (2018).
7. L. Novotny, M. R. Beversluis, K. S. Youngworth, and T. G. Brown, Phys. Rev. Lett. **86**, 5251 (2001).
8. R. Wang, C. Zhang, Y. Yang, S. Zhu, and X. C. Yuan, Opt. Lett. **37**, 2091 (2012).
9. M. Neugenbauer, P. Woźniak, A. Bag, A. Leuchs, and P. Banzer, Nat. Commun. **7**, 11286 (2016).
10. K. H. Kagawala, G. Di Guiseppe, A. F. Abouraddy, and B. E. A. Saleh, Nat. Photonics **7**, 72 (2013).
11. B. Ndagano, I. Nape, B. Perez-Garcia, S. Scholes, R. I. Hernandez-Aranda, T. Konrad, and A. Forbes, Nat. Phys. **13**, 397 (2017).
12. S. Mamani, L. Shi, T. Ahmed, R. Karnik, A. Rodríguez-Contreras, D. Nolan, and R. Alfano, J. Biophoton. **11**, e201800096 (2018).
13. M. A. Cox, C. Rosales-Guzmán, M. P. J. Lavery, D. J. Versfeld, and A. Forbes, Opt. Express **24**, 18105 (2016).
14. R. J. C. Spreeuw, Phys. Rev. A **63**, 062302 (2001).
15. C. E. R. Souza, J. A. O. Huguenin, P. Milman, and A. Z. Khoury, Phys. Rev. Lett. **99**, 160401 (2007).
16. C. V. S. Borges, M. Hor-Meyll, J. A. O. Huguenin, and A. Z. Khoury, Phys. Rev. A **82**, 033833 (2010).
17. A. F. Abouraddy, K. H. Kagawala, and B. E. A. Saleh, Opt. Lett. **39**, 2411 (2014).
18. S. Franke-Gold, J. Leach, M. J. Padgett, V. E. Lembessis, D. Ellinas, A. Wright, and A. S. Arnold, Opt. Express **15**, 8619 (2007).
19. S. A. Ponomarenko and G. P. Agrawal, Opt. Lett. **33**, 767 (2008).
20. H. E. Kondakci, M. A. Alonso, and A. F. Abouraddy, Opt. Lett. **44**, 2645 (2019).
21. P. Chowdhury, A. S. Majumdar, and G. S. Agarwal, Phys. Rev. A **88**, 013830 (2013).
22. S. Prabhakar, S. G. Reddy, A. Aadhi, C. Perumangatt, G. K. Samanta, and R. P. Singh, Phys. Rev. A **92**, 023822 (2015).
23. R. Simon and N. Mukunda, J. Opt. Soc. Am. A **10**, 95 (1993).
24. A. T. Friberg, B. Tervonen, and J. Turunen, J. Opt. Soc. Am. A **11**, 1818 (1994).
25. H. Wang, X. Peng, L. Liu, F. Wang, Y. Cai, and S. A. Ponomarenko, Opt. Lett. **44**, 3709 (2019).

26. S. A. Ponomarenko, Phys. Rev. E **64**, 036618 (2001).
27. S. A. Ponomarenko and G. P. Agrawal, Phys. Rev. E **69**, 036604 (2004).
28. Y. Cai and S. He, Appl. Phys. Lett. **89**, 041117 (2006).
29. F. Wang, Y. Cai, H. T. Eyyuboglu, and Y. Baykal, Opt. Lett. **37**, 184 (2012).
30. Y. Cai, Q. Lin, and O. Korotkova, Opt. Express **17**, 21472 (2009).
31. Z. Tong and O. Korotkova, Opt. Lett. **37**, 2595 (2012).
32. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, 1953), Vol. **I**.
33. U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University, 1997).
34. Y. Chen, A. Norrman, S. A. Ponomarenko, and A. T. Friberg, Prog. Opt. **65**, 105 (2020).
35. Z. Mei and O. Korotkova, Opt. Lett. **42**, 255 (2017).
36. F. Gori and M. Santarsiero, Opt. Lett. **43**, 595 (2018).
37. L. Wan and D. Zhao, Opt. Lett. **44**, 735 (2019).
38. C. Tian, S. Zhu, H. Huang, Y. Cai, and Z. Li, Opt. Lett. **45**, 5880 (2020).