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The diffusion of partially coherent beams in turbulent media

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Abstract

We study the spreading of the rms spatial and angular characteristics of partially coherent beams in turbulent media. The angular broadening of a beam is shown to be diffusion-like. The dynamics of the rms width of the beam is found to be determined by the interplay of free-space diffraction and turbulent diffusion. Our results indicate the conditions under which partially coherent beams are less sensitive to distortions caused by the atmospheric turbulence than are fully coherent beams. © 2002 Published by Elsevier Science B.V.

1. Introduction

The propagation of light in turbulent media in the multiple scattering regime has been of considerable interest in connection with many fundamental [1] as well as applied topics such as, for example, weaponry, optical imaging, and satellite communications [2].

To date, several alternative methods for treating the propagation of beams in turbulent media have been proposed [2–4]. It was shown in [5] that as far as the second-order field correlations are concerned, all of these approaches are equivalent. In order to successfully apply any of these methods to realistic situations, one has to assume either Gaussian statistics of the refractive index fluctuations or the lognormal statistics of the optical field. However, the experiments over long propagation paths have indicated that lognormal model is inappropriate in the strong fluctuation regime [6]. Therefore, an altogether different approach may be necessary to study how the characteristics of beams are affected by strong turbulence over sufficiently large propagation distances. Another open question is the influence of the state of coherence of the incident beam on its subsequent propagation in turbulent media. Preliminary numerical results [7] suggest that partially coherent beams might be less affected by the atmospheric turbulence than are their fully coherent counterparts.

In this paper, we develop a theory that makes it possible to calculate the spatial and angular spreads of partially coherent beams propagating in a turbulent medium under all conditions of turbulence. Our approach employs a formal analogy

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between quantum mechanics in Hilbert space and paraxial wave optics. The advantages of such an approach are threefold. First of all, it clarifies the diffusive nature of the beam spreading in turbulent media, and it helps to identify the key parameters determining the behavior of the beam: the characteristic length associated with the spreading of the beam due to diffraction and the turbulent diffusion length. Secondly, our approach does not require any assumptions regarding the type of statistics of the refractive index fluctuations. Finally, the analysis of our analytic expressions for the rms width and the angular spread of the beam enables us to find conditions under which the distorting influence of a turbulent medium is less pronounced for partially coherent beams than it is for fully coherent ones.

2. Hilbert space theory of beam propagation in turbulent media

We begin by recalling that as long as backscattering and depolarization effects are negligible, the optical field $U(\boldsymbol{\rho},z)\ \exp(-\mathrm{i}\omega t)$ of a paraxial beam, propagating in the z-direction into the half-space $z>0$, obeys the scalar parabolic equation [2, Section 5.7]:

$$2\mathrm{i}k\partial_z U(\boldsymbol{\rho},z)+\nabla_\perp^2 U(\boldsymbol{\rho},z)+2k^2 n_1(\boldsymbol{\rho},z)U(\boldsymbol{\rho},z)=0, \tag{1}$$

where $k=\omega/c$, $n_1(\boldsymbol{\rho},z)$ is the fluctuation of the refractive index, and $\nabla_\perp$ is the gradient in a plane transverse to the direction of propagation of the beam. The first- and second-order statistical properties of the fluctuation of the refractive index $n_1(\boldsymbol{\rho},z)$ are given by the expressions [3, Section 65]

$$\langle n_1(\boldsymbol{\rho},z)\rangle_n=0, \tag{2a}$$

and

$$\langle n_1(\boldsymbol{\rho}_1,z_1)n_1(\boldsymbol{\rho}_2,z_2)\rangle_n=\delta(z_1-z_2)B_n(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2), \tag{2b}$$

where $\langle\cdots\rangle_n$ denote the average over the ensemble of realizations of the refractive index fluctuations $n_1(\boldsymbol{\rho},z)$, and Eq. (2b) implies Markovian character of the refractive index fluctuations in the longitudinal direction, an approximation whose validity improves with the propagation distance [3, Section 65]. Further, assuming that $n_1(\boldsymbol{\rho},z)$ is statistically homogeneous, we may introduce the spectral density $\phi_n(\mathbf{k}_\perp)$ of the refractive index fluctuations in a transverse plane. As a consequence of spatial analogue of the Wiener–Khintchine theorem [8, Section 2.4], the spectral density $\phi_n(\mathbf{k}_\perp)$ is related to the second-order correlation function B_n of $n_1(\boldsymbol{\rho},z)$ by a Fourier transform:

$$B_n(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)=\int \mathrm{d}^2k_\perp \mathrm{e}^{\mathrm{i}k_\perp(\rho_1-\rho_2)}\phi_n(\mathbf{k}_\perp). \tag{3}$$

We will now employ a formal analogy which exists between paraxial wave optics in free space and quantum mechanics [9–11] to formulate the problem of beam propagation in turbulent media in the language of quantum mechanics in Hilbert space. To this end, we consider the optical field $U(\boldsymbol{\rho},z)$ to be the product of the "bra" vector $\langle\boldsymbol{\rho}|$ and the "ket" vector of the state $|U(z)\rangle$ of a beam at the distance z:

$$U(\boldsymbol{\rho},z)=\langle\boldsymbol{\rho}|U(z)\rangle. \tag{4}$$

Next we introduce the statistical operator of the incident beam by the expression

$$\widehat{W}(0)=\overline{|U(0)\rangle\langle U(0)|}, \tag{5}$$

where $|U(0)\rangle$ is a "ket" vector of the state of the beam at $z=0$, and the overbar denotes the average over the ensemble of statistical realizations of the incident beams. A matrix element of the statistical operator in coordinate representation is the cross-spectral density $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)$ of the beam at a pair of points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ in the plane $z=0$, i.e.,

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)\equiv\langle U^*(\boldsymbol{\rho}_1,0)U(\boldsymbol{\rho}_2,0)\rangle = \langle\boldsymbol{\rho}_1|W|\boldsymbol{\rho}_2\rangle. \tag{6}$$

The phase-space dynamics of the beam is characterized by the radius and by the angular spread of the beam. To describe the latter, one can represent the beam as a superposition of plane waves. A plane wave with the transverse wave vector $\mathbf{k}_\perp$ propagates in a direction which makes the angle θ with the z-axis given by

$$\tan\theta=|\mathbf{k}_\perp|/k_z. \tag{7}$$

For a paraxial beam, one has $\tan\theta\approx\sin\theta\approx\theta$, and it follows from definition (7) that $\theta\simeq|\mathbf{k}_\perp|/k$.

Let us introduce the Hilbert-space radius operator $\hat{\rho}$ and the angle operator $\hat{\theta}$. The latter is defined by the expression

$$\hat{\theta} \equiv \hat{k}_{\perp}/k. \tag{8}$$

In coordinate representation, the angle operator is given by the expression

$$\hat{\theta} = -(\mathrm{i}/k)\nabla_{\perp}. \tag{9}$$

It follows at once from Eq. (5) that the expectation value of any function $f(\hat{\rho}, \hat{\theta})$ in the transverse plane $z = 0$ can be evaluated according to the rule

$$\overline{f(\hat{\rho}, \hat{\theta})} = \mathrm{Tr}[f(\hat{\rho}, \hat{\theta})W]/\mathrm{Tr}[W]. \tag{10}$$

It can be readily verified that Eq. (10) reduces, in particular cases, to the usual definitions of averages over distributions of the optical intensity in the plane $z = 0$ and the radiant intensity in coordinate and momentum representations, respectively. Consider, for example, the rms width of the incident beam. On converting Eq. (10) to coordinate representation, one finds that

$$\overline{\hat{\rho}^2} = \frac{\int \mathrm{d}^2\rho \rho^2 W(\boldsymbol{\rho}, \boldsymbol{\rho}, 0)}{\int \mathrm{d}^2\rho W(\boldsymbol{\rho}, \boldsymbol{\rho}, 0)}, \tag{11}$$

where $W(\boldsymbol{\rho}, \boldsymbol{\rho}, 0) \equiv I(\boldsymbol{\rho}, 0)$ is the intensity of the incident beam in the plane $z = 0$. Similarly, using momentum representation, we obtain for the mean-square angular spread of the incident beam the expression

$$\overline{\hat{\theta}^2} = \frac{\int \mathrm{d}^2\theta \theta^2 \widetilde{W}(-k\boldsymbol{\theta}, k\boldsymbol{\theta}, 0)}{\int \mathrm{d}^2\theta \widetilde{W}(-k\boldsymbol{\theta}, k\boldsymbol{\theta}, 0)}. \tag{12}$$

Here $\widetilde{W}(\mathbf{f}_1, \mathbf{f}_2, 0)$ is a four-dimensional Fourier transform of the cross-spectral density of the incident beam, defined as

$$\widetilde{W}(\mathbf{f}_1, \mathbf{f}_2, 0) = \int \frac{\mathrm{d}^2\rho_1}{(2\pi)^2} \int \frac{\mathrm{d}^2\rho_2}{(2\pi)^2} W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, 0) \times \mathrm{e}^{\mathrm{i}(\mathbf{f}_1\cdot\boldsymbol{\rho}_1 + \mathbf{f}_2\cdot\boldsymbol{\rho}_2)}. \tag{13}$$

In obtaining Eq. (12), we have made use of the connection between coordinate and momentum representations viz.

$$\langle \boldsymbol{\theta} | \boldsymbol{\rho} \rangle = \frac{1}{(2\pi k)^2} \exp(\mathrm{i}k\boldsymbol{\theta} \cdot \boldsymbol{\rho}). \tag{14}$$

Eqs. (11) and (12) provide a connection between the radius $\hat{\rho}$ and the angle $\hat{\theta}$ operators on one hand and the rms spatial and angular spreads of the beam on the other.

In Hilbert space, Eq. (1) takes the form of the Schrödinger equation

$$\mathrm{i}k\partial_z|U(z)\rangle = \hat{H}|U(z)\rangle, \tag{15a}$$

where the Hamiltonian $\hat{H}$ is given by the expression

$$\hat{H} = k^2[\hat{\theta}^2/2 - n_1(\hat{\rho}, z)]. \tag{15b}$$

Further, the radius and angle operators form a conjugate pair with the commutator:

$$[\hat{\rho}, \hat{\theta}] = \mathrm{i}/k. \tag{16}$$

Eq. (16) expresses the well-known diffraction limit of the beam localization in free space. It should be mentioned that as $\lambda \equiv 2\pi/k \to 0$ the commutator vanishes, which corresponds to the limit of geometric optics.

3. The rms spatial and angular spreads of a beam

Eqs. (15a) and (15b) specify the state of the beam in any transverse plane $z = \mathrm{const} > 0$ given the state of the incident beam in the plane $z = 0$. Alternatively, one can study the beam dynamics in the Heisenberg picture, where, as is well-known [12, Chapter 2], one follows the evolution of Heisenberg operators, while the state of the system remains unchanged. The evolution of the rms spatial and angular spreads of the beam with the propagation distance is more conveniently described in the Heisenberg picture. The equations of motion for the operators $\hat{\rho}$ and $\hat{\theta}$ are

$$\mathrm{i}k\partial_z\hat{\rho} = [\hat{\rho}, \hat{H}], \tag{17a}$$

$$\mathrm{i}k\partial_z\hat{\theta} = [\hat{\theta}, \hat{H}]. \tag{17b}$$

Using Eqs. (15b) and (16), as well as the relation,

$$[\hat{\theta}, n_1(\hat{\rho}, z)] = -(\mathrm{i}/k)\nabla_{\perp}n_1(\hat{\rho}, z), \tag{18}$$

one can formally integrate Eqs. (17a) and (17b) with the result that

$$\hat{\rho}(z) = \hat{\rho}(0) + \int_0^z \mathrm{d}\xi \hat{\theta}(\xi), \tag{19a}$$

$$\hat{\theta}(z) = \hat{\theta}(0) + \int_0^z \mathrm{d}\eta \nabla_\rho n_1[\hat{\rho}(\eta), \eta]. \tag{19b}$$

It follows from Eq. (2a) that turbulence does not affect the dynamics of the first-order moments of $\hat{\rho}$ and $\hat{\theta}$. We assume, for simplicity, that the incident beam is circularly symmetric in the transverse plane, implying that $\langle\hat{\theta}(0)\rangle = 0$ and $\langle\hat{\rho}(0)\rangle = 0$, where the angle brackets denote the combined average over the ensembles of the incident beams and of the atmospheric fluctuations. One can then infer from Eqs. (2a) and (19a), (19b) that the first-order moments of the radius and the angle vanish in any plane $z = \text{const}$ in the half-space $z > 0$. The calculation of the second-order moments leads to the expressions

$$\langle\hat{\theta}^2(z)\rangle = \overline{\hat{\theta}^2(0)} + I_1(z), \tag{20a}$$

$$\langle\hat{\rho}^2(z)\rangle = \overline{\hat{\rho}^2(0)} + \overline{\{\hat{\rho}(0), \hat{\theta}(0)\}}z + \overline{\hat{\theta}^2(0)}z^2 + I_2(z). \tag{20b}$$

Here {,} stands for the anticommutator of a pair of operators, and

$$I_1(z) = \int_0^z \mathrm{d}\eta \int_0^z \mathrm{d}\eta' \nabla_\rho \nabla_{\rho'} \langle n_1[\tilde{\rho}(\eta), \eta] n_1[\tilde{\rho}(\eta'), \eta']\rangle_n, \tag{21}$$

$$I_2(z) = \int_0^z \mathrm{d}\xi \int_0^z \mathrm{d}\xi' \int_0^\xi \mathrm{d}\eta \int_0^{\xi'} \mathrm{d}\eta' \nabla_\rho \nabla_{\rho'} \times \langle n_1[\tilde{\rho}(\eta), \eta] n_1[\tilde{\rho}(\eta'), \eta']\rangle_n. \tag{22}$$

In deriving the expressions for $I_1(z)$ and $I_2(z)$ in Eqs. (20a) and (20b) we have made a semiclassical approximation [12, Section 2.4] by replacing the operators $\hat{\rho}$ by the c – numbers $\tilde{\rho}$ on the right-hand sides of Eqs. (21) and (22). This assumption amounts to neglecting contributions of the order of the commutator of the radius and angle operators. It follows at once from Eq. (16) that this is equivalent to using the geometric optics approximation, $\lambda \to 0$. There are two reasons that justify the use of such an approximation. First, the operator nature of $\hat{\rho}(z)$ results in the noncommutativity of the radius operators in different transverse planes z and z', $[\hat{\rho}(z), \hat{\rho}(z')] \neq 0$. However, due to the Markovian character of the refractive index fluctuations in the longitudinal direction, there is no contribution to the correlation function of $n_1(\boldsymbol{\rho}, z)$ from points in different transverse planes z and z'. Secondly, the refractive index fluctuations $n_1(\boldsymbol{\rho}, z)$ vary slowly at the spatial scales of the order of the wavelength. Consequently, $|n_1(\hat{\rho}, z) - n_1(\tilde{\rho}, z)| \sim |n_1|\lambda/l_0 \ll |n_1|$, where the inner scale of the refractive index fluctuations l_0 is typically of the order $l_0 \sim 1$ cm, whereas $\lambda \sim 5 \times 10^{-5}$ cm.

The averaging in Eqs. (21) and (22) can then be performed with the help of Eq. (2b), and after straightforward integrations, one obtains the expressions

$$I_1(z) = -z\nabla_\rho^2 B_n(0), \tag{23a}$$

$$I_2(z) = (-z^3/2)\nabla_\rho^2 B_n(0). \tag{23b}$$

Using the definition (3) of the spectral density $\phi_n(\mathbf{k}_\perp)$ of the refractive index fluctuations, it can be readily demonstrated that

$$-\nabla_\rho^2 B_n(0) = 2\pi \int \mathrm{d}^2 k_\perp k_\perp^2 \phi_n(\mathbf{k}_\perp). \tag{24}$$

On substituting from Eqs. (23a) and (23b) and taking into account Eq. (24), we obtain for the rms angular and spatial spreads of the beam the expressions

$$\langle\hat{\theta}^2(z)\rangle = \hat{\theta}_0^2 + 2\mathscr{D}_t z, \tag{25a}$$

and

$$\langle\hat{\rho}^2(z)\rangle = \hat{\rho}_0^2 + \overline{\{\hat{\rho}_0, \hat{\theta}_0\}}z + \hat{\theta}_0^2 z^2 + \mathscr{D}_t z^3, \tag{25b}$$

where $\theta_0^2 \equiv \overline{\theta(0)^2}$, and $\rho_0^2 \equiv \overline{\rho(0)^2}$. Eq. (25a) indicates that the angular spreading of the beam is of a diffusive type with the diffusion coefficient $\mathscr{D}_t$, which is given by the expression

$$\mathscr{D}_t \equiv 2\pi^2 \int_0^\infty \mathrm{d}k_\perp k_\perp^3 \phi_n(k_\perp). \tag{26}$$

Here it was assumed, for simplicity, that the random medium is statistically isotropic.

It follows at once from Eq. (10) that, in complete analogy with quantum mechanics [8, Section 11.8.1], the expectation of the Weyl-ordered product of operators $\{\hat{\rho}_0, \hat{\theta}_0\}$ can be expressed as

$$\overline{\{\hat{\rho}_0, \hat{\theta}_0\}} = 2\frac{\int \mathrm{d}^2\rho \int \mathrm{d}^2\theta \rho\theta \mathscr{W}(\boldsymbol{\rho}, \boldsymbol{\theta})}{\int \mathrm{d}^2\rho \int \mathrm{d}^2\theta \mathscr{W}(\boldsymbol{\rho}, \boldsymbol{\theta})}, \tag{27}$$

where $\mathscr{W}(\boldsymbol{\rho}, \boldsymbol{\theta})$ is the Wigner function, defined by the expression

$$\mathscr{W}(\mathbf{R}, k_\perp) = \int \mathrm{d}^2\rho \langle \mathbf{R} - \boldsymbol{\rho}/2 | W | \mathbf{R} + \boldsymbol{\rho}/2 \rangle \mathrm{e}^{-\mathrm{i}k_\perp \boldsymbol{\rho}}. \tag{28}$$

It should be noted that Eq. (25b) has the same structure as previously reported results in [13] and [14] for the cases of fully coherent and of partially coherent incident beams, respectively. There is however a difference in the numerical value of the diffusion coefficient $\mathscr{D}_t$. This difference may be attributed to approximations made in [13,14] regarding the statistics of the medium fluctuations.

The diffusion nature of angular spreading of the beam in the turbulent atmosphere, which is expressed by Eq. (25a), can be also understood within the framework of the random walk model [15]. In this model, a beam is represented as a collection of particles which are sometimes inappropriately referred to as photons. They are scattered from the spatial fluctuations of the medium. In each scattering event, the angle that the direction of propagation of such a particle makes with the mean direction of propagation of the beam is changed by the amount $\Delta\theta$. In the turbulent atmosphere, the probability density of the angular scattering of the particles is highly peaked in the forward direction and consequently $\Delta\theta \ll 1$. Since the scattering events are independent, the overall rms angular spread of the beam is of the order of $n\Delta\theta$, where n is the number of scattering events. The latter quantity can be estimated as $n \sim z/l_0$, where z is a propagation distance and l_0 is a mean free path of the particle. It follows from these heuristic arguments that the rms angular spread of the beam is proportional to the propagation distance z which is a signature of the diffusive behavior.

4. Beam characterization in turbulent media

It readily follows from Eqs. (25a) and (25b) that at long propagation distances, the turbulence takes its toll on any incident beam, leading to an additional significant broadening of the rms characteristics of the beam. However, so long as the propagation distance is sufficiently short, it is possible to control the phase-space properties of beams propagating in the turbulent atmosphere by a choice of their initial state of spatial coherence. The corresponding optimization problem depends on a particular application. For instance, in some applications it is required to have a beam with the smallest product of the rms width and the angular spread, i.e, the smallest "M^2 quality product" [16], in a given transverse plane z. It follows from Eqs. (25a) and (25b) that to achieve this goal, one has to launch an incident beam with the minimum M^2 product as well. It is known [11] that the minimum of M^2 is attained by a fully spatially coherent Gaussian beam. On the other hand, for applications such as targeting and satellite communications, it may be desirable to minimize the spreading due to diffraction and turbulence of the initial "targeting" parameters of a beam. For instance, one may wish to use beams whose angular spreads relative to their initial values are sufficiently small.

The analysis of Eqs. (25a) and (25b) indicates that there are two characteristic spatial scales which determine the influence of diffraction and turbulence. A typical diffraction scale, the well-known Rayleigh range [17,18], is defined as the distance at which the mean-square radius of the incident beam increases by a factor of two due to diffraction. For a quasi-monochromatic beam, it is given by the expression

$$z_R = \rho_0/\theta_0. \tag{29}$$

Here we have assumed, for simplicity, that the Wigner function is an even function of its arguments so that the term proportional to the anti-commutator of the angle and the radius operators vanishes identically. The other length scale, a typical diffusion length $\mathscr{D}_t$, is defined as the distance at which the mean-square angular spread is twice as large as its initial value:

$$z_D = \theta_0^2/2\mathscr{D}_t. \tag{30}$$

Using the definitions (29) and (30), one can rewrite the equations for the rms width and the relative rms angular spread of the beam, Eqs. (25a) and (25b), in the dimensionless form

$$\frac{\Delta\theta(z)}{\theta_0} = \sqrt{\frac{z}{z_D}}, \tag{31a}$$

$$\frac{\rho(z)}{\rho_0} = \sqrt{1 + \frac{z^2}{z_R^2} + \frac{z^3}{2 z_D z_R^2}}. \tag{31b}$$

Here we have introduced the notations $\rho^2(z) \equiv \langle \hat{\rho}^2(z) \rangle$, and $\theta^2(z) \equiv \langle \hat{\theta}^2(z) \rangle$ as well as $\Delta\theta(z) \equiv (\theta^2(z) - \theta_0^2)^{1/2}$. It follows from Eq. (31a) that the longer the diffusion length, the shorter the relative rms angular spread of the beam at a given distance z. Since the diffusion length increases with the initial angular spread, one can minimize the relative rms angular spread by maximizing the initial angular spread of the beam. On the other hand, it is seen from Eq. (29) that a larger initial angular spread of the beam results in a shorter Rayleigh range. The optimal initial angular spread of the beam therefore corresponds to the case when the diffusion length is of the same order of magnitude as the Rayleigh range. It can be inferred from Eqs. (31a) and (31b) that under these circumstances, both the rms radius and the relative rms angular spread do not significantly increase after the beam has passed a distance of the order of $z \sim z_D \sim z_R$. Consequently, one can still use such beams in targeting or communications applications at distances of this order of magnitude.

Let us estimate typical values of the Rayleigh range and the diffusion length of incident fully spatially coherent beams. To this end, we consider the Tatarskii model for the spectral density of the refractive index fluctuations [2, Section 3.3.2]

$$\phi_n(k) = 0.033 C_n^2 k^{-11/3} e^{-k^2 l_0^2}, \tag{32}$$

where C_n is a so-called structure constant, and l_0 is the inner scale of the turbulence. Under typical conditions of turbulence, $C_n \sim 0.3 \times 10^{-14}\ \mathrm{cm}^{-2/3}$, and $l_0 \sim 1$ cm. On substituting from Eq. (32) into Eq. (26) and on evaluating the integral, one obtains the estimate $\mathscr{D}_t \simeq 0.5 \times 10^{-14}\ \mathrm{cm}^{-1}$. The angular spread of an incident fully coherent beam of the rms width ρ_0 is of the order of

$$\theta_0 \simeq \lambda/\rho_0. \tag{33}$$

On substituting this expression for θ_0 into Eqs. (29) and (30), we arrive at the expression for the rms width of the beam for which the Rayleigh range is approximately equal to the diffusion length,

$$\rho_0 \simeq (\lambda^3/2\mathscr{D}_t)^{1/4}, \tag{34}$$

which, at optical wavelengths $\lambda \sim 5 \times 10^{-5}$ cm leads to $\rho_0 \sim 2$ cm. The Rayleigh range corresponding to this rms width is $z_R \sim 1$ km.

Let us now consider a quasi-homogeneous source [8, Section 5.2.2] generating a beam whose spectral coherence length ρ_c is much smaller than the size of the source, R_0, i.e., $\rho_c \ll R_0$. The initial angular spread of such a beam can be estimated as

$$\theta_0 \simeq \lambda/\rho_c. \tag{35}$$

It follows from Eqs. (29), (30) and (35) that the condition $z_R \simeq z_D$ is satisfied for the partially coherent beam provided that

$$\rho_c \simeq \left(\frac{\lambda^3}{2\mathscr{D}_t R_0} \right)^{1/3}. \tag{36}$$

Assuming $R_0 \sim 10^2$ cm, one obtains from Eq. (36) that $\rho_c \sim 0.5$ cm. It follows that the Rayleigh range as well as the diffusion length for such a partially coherent beam is $z_R \simeq z_D \sim 10$ km. One can infer from these numerical examples that beams generated by quasi-homogeneous sources might indeed propagate over longer distances in the atmosphere without significant spreading than do fully coherent beams.

5. Numerical examples

The analysis and conclusions of the previous section apply to all partially coherent beams regardless of a particular functional form of their cross-spectral density. We will now illustrate these predictions by an example. Consider a Gaussian–Schell model source producing a partially coherent beam with the cross-spectral density which is given by the expression [8, Section 5.3.2]

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, 0) \propto \exp\left(-\frac{\boldsymbol{\rho}_1^2 + \boldsymbol{\rho}_2^2}{4\sigma_I^2}\right) \exp\left[-\frac{(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2}{2\sigma_c^2}\right], \tag{37}$$

where σ_I and σ_c are the rms width and the spectral coherence length in the source plane, respectively.

On substituting from Eq. (37) into Eq. (12) and on performing straightforward integrations, one obtains for the initial angular spread θ_0 the expression

$$\theta_0 = \sqrt{2}/k\sigma_{\text{eff}}, \tag{38}$$

where

$$\frac{1}{\sigma_{\text{eff}}^2} = \frac{1}{\sigma_c^2} + \frac{1}{4\sigma_I^2}. \tag{39}$$

It follows from Eq. (31a) and (31b) that in the absence of turbulence, the rms angular spread of the beam would be an integral of motion. Consequently, the influence of turbulence is most clearly revealed by studying angular spreading of the beam. On substituting from Eqs. (38) and (39) into Eqs. (25a) and (31a), we determine the rms angular spread as well as the rms angular spread relative to its value at $z = 0$. The results are displayed in Figs. 1 and 2. In Fig. 1. the absolute value of the rms angular spread is shown as a function of the propagation distance z for incident beams with the same value of σ_I, but with different values of the spectral coherence length σ_c. It is seen from the figure that the angular spread slowly increases with decreasing σ_c. On the other hand, Fig. 2 indicates a pronounced decrease of the relative rms angular spread of the beam with decrease of the coherence length of the incident beam.

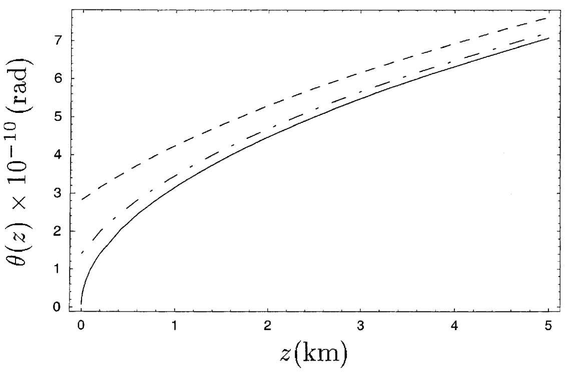

Fig. 1. The rms angular spread of a partially coherent beam with the width $\sigma_I = 10$ cm at the waist, for different values of the spectral coherence length σ_c. Solid line: $\sigma_c = 100$ cm, long-dashed line: $\sigma_c = 1$ cm, and short-dashed line: $\sigma_c = 0.5$ cm.

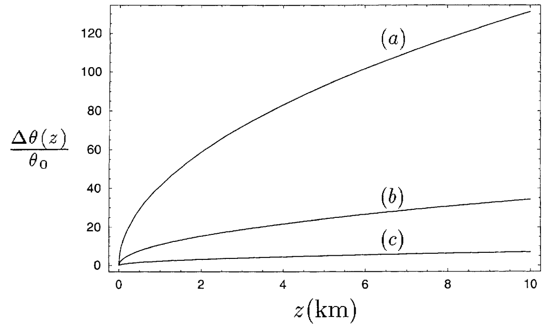

Fig. 2. The rms angular spread relative to its initial value θ_0 at $z = 0$ for a partially coherent beam with the width $\sigma_I = 10$ cm at the waist, for different values of the spectral coherence length σ_c: (a) $\sigma_c = 50$ cm, (b) $\sigma_c = 5$ cm (c) $\sigma_c = 1$ cm.

6. Conclusions

We have derived analytic expressions for the rms width and the rms angular spread of partially coherent beams propagating in turbulent media. The contribution to the angular spreading caused by turbulence has been demonstrated to be of a diffusion type. The phase-space dynamics of any beam has been shown to depend on the relation between the Rayleigh range and the diffusion length which characterize the influence of diffraction and turbulence, respectively. We have also shown that in order to minimize the broadening of the beam, the state of spatial coherence of the incident beam should be reduced. In particular, if a fully coherent source of the size ρ_0 is converted into a quasi-homogeneous source of the effective size R_0 and with the spectral coherence length ρ_c such that $\rho_c \leqslant \rho_0 \ll R_0$, a beam generated by such a quasi-homogeneous source can propagate over a considerably longer distance without significant spreading than can a beam which is generated by the fully coherent source of the effective. In practice, such quasi-homogeneous sources can be produced by forming suitable arrays of uncorrelated sources or by the use of a rotating diffuser, for example.

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