Energy spectrum of a nonstationary ensemble of pulses

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We introduce a new definition of the energy spectrum of a nonstationary ensemble of pulses that reduces to the usual ones in the limit of statistically stationary ensembles of signals and of fully temporarily coherent ensembles.

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Fluctuations of light fields have been the subject of extensive studies in the context of optical coherence theory.[1] One of the key aspects of the theory is the spectral decomposition of light generated by a source of any state of coherence. To date, however, most of the work on this subject has been primarily concerned with light that is either statistically stationary or cyclostationary.[2–4] The spectrum of such fields can be obtained from the Wiener–Khintchine theorem (Ref. 1, Sec. 2.4). On the other hand, optical pulses are, by their nature, nonstationary, and the definition of the spectrum of such pulses is an important issue that has so far received relatively little attention (see, however, Refs. 5–10). The whole field of ultrafast phenomena suffers from this problem, and it is common to assume that pulses are initially coherent and remain fully coherent while propagating inside linear or nonlinear optical media such as optical fibers.[11]

The first attempts to introduce spectra of nonstationary light[5,6] resulted in some definitions of a nonstationary power spectrum. Unfortunately, such spectra are not necessarily positive. To correct this shortcoming, Mark[7] proposed a convolution procedure with a "window" function that yields a nonnegative spectrum. Later, Eberly and Wódkiewicz[8] discussed the physical meaning of such a convolution procedure and showed that the instantaneous power spectrum, which they called the "physical spectrum" of light, is directly related to photodetection measurements. However, this "physical spectrum" of a pulse depends not only on the frequency but also on time. This feature of the "physical spectrum" does not agree with the intuitive notion of the spectrum as the energy distribution of a pulse as a function of the frequency, which is, in some sense, complementary to the energy distribution of the pulse in the time domain.[12]

In the present Letter, we introduce a definition of the time-independent energy spectrum of a nonstationary ensemble of pulses that is in agreement with such an intuitive view of this concept. We also show that, for short pulses, such an energy spectrum can, in principle, be measured by square-law detectors.

In an attempt to introduce a satisfactory definition of the energy spectrum of a nonstationary ensemble of pulses, we will be guided by a simple physical picture of the spectrum as the distribution of energy of a pulse over monochromatic components. We begin by considering a statistical realization of an optical pulse $U(t)$, t denoting the time, which is assumed to propagate either in free space or in medium. Let

$$\tilde{U}(\omega) \equiv \int_{-\infty}^{+\infty} \mathrm{d}t U(t) \exp(-i\omega t), \tag{1}$$

be a Fourier transform of $U(t)$. We introduce a nonnegative quantity $S(\omega)$ by the expression

$$S(\omega) \equiv \langle |\tilde{U}(\omega)|^2 \rangle, \tag{2}$$

where the angle brackets denote the average over the ensemble of realizations of the pulse. On substituting from Eq. (1) into Eq. (2) and interchanging the order of ensemble averaging and of integrations, we obtain the following expression for $S(\omega)$:

$$S(\omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \mathrm{d}t_2 \Gamma(t_1, t_2) \exp[i\omega(t_1 - t_2)], \tag{3}$$

where

$$\Gamma(t_1, t_2) = \langle U^*(t_1) U(t_2) \rangle \tag{4}$$

is the second-order correlation function of the pulse at times t_1 and t_2.

It follows at once from Eq. (2) and from Parseval's theorem on Fourier transforms[13] that

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega S(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}t \langle |U(t)|^2 \rangle. \tag{5}$$

The quantity on the right-hand side of Eq. (5) is equal to the total energy carried by the signal. It is seen from Eqs. (2) and (5) that the nonnegative quantity $S(\omega)$, which represents the distribution of energy of the pulse in frequency domain, may be taken as a definition of the energy spectrum. Even though formally our energy spectrum can be obtained from the so-called two-frequency spectrum introduced in Ref. 9, the underlying physical picture is quite different. While the physical meaning of the two-frequency spectrum is somewhat obscure, the energy spectrum defined by Eq. (2) has a clear physical meaning.

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Moreover, we will show below how the energy spectrum is related to photocount measurements with square-law detectors.

Let us briefly examine some limiting cases. First we consider statistically stationary signals. Such signals are, of course, an idealization, because no optical signal that carries a finite amount of energy can be strictly stationary. At best, it can be quasi-stationary. The second-order correlation function of a quasi-stationary signal has the form[14]

$$\Gamma(t_1, t_2) = \Phi\left(\frac{t_1 + t_2}{2}\right)\xi(t_1 - t_2), \tag{6}$$

where $\Phi(t)$ is a "slow function" of t and $\xi(t)$ is a "fast function" of t in the sense that a characteristic rate of change of the latter is much greater than that of the former. On substituting from Eq. (6) into Eq. (3), one finds that the energy spectrum is a Fourier transform of the "stationary part" ξ of the correlation function Γ, a result that is an analog of the Wiener–Khintchine theorem for stationary processes.

If the pulse is temporarily coherent, a characteristic temporal width T_p of its intensity profile is much smaller than its typical coherence time τ_c, which may be defined by the expression

$$\tau_c = \left[\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}t_1 \mathrm{d}t_2 (t_1 - t_2)^2 |\gamma(t_1, t_2)|^2}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}t_1 \mathrm{d}t_2 |\gamma(t_1, t_2)|^2}\right]^{1/2}. \tag{7}$$

Consequently, the degree of temporal coherence of such pulses, which may be defined by the expression

$$\gamma(t_1, t_2) \equiv \frac{\Gamma(t_1, t_2)}{\sqrt{I(t_1)}\sqrt{I(t_2)}}, \tag{8}$$

where $I(t) = (t, t) = \langle U^*(t)U(t)\rangle$ is essentially unimodular, i.e., $|\gamma(t_1, t_2)| \simeq 1$, for any pair of time arguments t_1 and t_2 except those that are associated with the tails of the pulse and thus carry a negligible amount of energy. It follows at once that the degree of temporal coherence of fully coherent pulses can be expressed in the form

$$\gamma(t_1, t_2) = \exp[i\phi(t_1, t_2)]. \tag{9}$$

It can be shown, following the method of Ref. 1 (Sec. 4.5.2; see also Ref. 15) that Eq. (9), together with Hermiticity and the nonnegative definiteness of the temporal degree of coherence, implies that in this case, $\gamma(t_1, t_2)$ is necessarily of the form

$$\gamma(t_1, t_2) = \exp\{i[\psi(t_1) - \psi(t_2)]\}. \tag{10}$$

It readily follows from Eqs. (8) and (10) that the second-order correlation function of a fully coherent pulse factorizes in the form

$$\Gamma(t_1, t_2) = \Psi^*(t_1)\Psi(t_2), \tag{11}$$

where $\Psi(t) = \sqrt{I(t)}\exp[-i\psi(t)]$. Consequently, the energy spectrum, defined by Eq. (3), of a fully coherent pulses is just the squared modulus of a Fourier transform of the pulse, in agreement with the usual definition of such a spectrum.[10]

We will now consider the relation between the energy spectrum and the so-called "physical spectrum" of Ref. 8. The latter is defined, up to a proportionality constant, as an instantaneous photocounting rate $\mathcal{R}$ of a somewhat idealized square-law detector

$$\mathcal{R}(t) = \langle |U_D(t)|^2 \rangle. \tag{12}$$

Here

$$U_D(t) = \int_{-\infty}^{+\infty} \mathrm{d}t' H(t - t') U(t') \exp[-i\omega_f (t - t')], \tag{13}$$

represents an optical signal that reaches a photodetector having passed through a tunable filter centered at frequency ω_f. In Eq. (13), $H(t)$ is a casual response function of the filter, assumed to be normalized so that $\int_{-\infty}^{+\infty} \mathrm{d}t |H(t)|^2 = 1$. It follows at once from Eqs. (12) and (13) that the "physical spectrum" is equal to

$$\mathcal{R}(t, \omega_f) = \int_{-\infty}^{+\infty} \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}t'' H^*(t - t'') H(t - t') \times \Gamma(t_1, t_2) \exp[-i\omega_f (t' - t'')]. \tag{14}$$

It should be noted that the precise form of the response function in Eq. (14) is not important as long as it is casual and normalizable. In particular, if we choose a Fabry–Perot interferometer as the filter, the normalized response function has the form

$$H(t) = \sqrt{2\gamma_f}\, \theta(t) \exp(-\gamma_f t), \tag{15}$$

where $\gamma_f \ll \omega_f$ is the bandwidth of the response function of the interferometer and $\theta(t)$ is the unit step function.

To transform from the power spectrum to the energy spectrum, we introduce a time-integrated spectrum $\overline{S}(\omega_f)$ that, up to a proportionality constant, is given by the total number of counts recorded by a photodetector in a time interval $2T_0$ (Ref. 16):

$$\overline{S}(\omega_f) = \int_{-T_0}^{T_0} \mathrm{d}t \mathcal{R}(t, \omega_f). \tag{16}$$

If the counting interval is sufficiently long that almost all of the pulse energy is detected, we can obtain, with the help of Eqs. (14)–(16) and Eq. (2) for the time-integrated spectrum $\sigma(\omega_f)$, the expression

$$\sigma(\omega_f) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\gamma_f}{{\gamma_f}^2 + (\omega - \omega_f)^2} S(\omega). \tag{17}$$

In the deviation of Eq. (17) we have assumed that $T_0 >> \max(1/\gamma_f, T_{\mathrm{eff}})$, where $1/T_{\mathrm{eff}}$ is an effective width of the energy spectrum of the pulse, and hence, the limits of integration in Eq. (17) were extended from $-\infty$ to ∞. We can conclude from Eq. (17) that the spectrum that is actually measured in photodetection experiments is a smoothed version of the theoretical energy spectrum given by Eq. (3). Moreover, for sufficiently short pulses, $\gamma_f \ll 1/T_{\mathrm{eff}}$, and hence the

spectral transmission function of the filter can then be well approximated by the delta function. As a result, our theoretical energy spectrum, defined by Eq. (3), becomes indistinguishable from $\sigma(\omega_f)$.

In summary, we have introduced a definition of the spectrum of a nonstationary ensemble of pulses, and we have elucidated the connection between the energy spectrum and the spectrum obtained from photocount measurements by square-law detectors. Although we discussed the energy spectrum of optical pulses, such a concept is equally applicable to nonstationary signals of any kind.

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