Correlation matrix of a completely polarized, statistically stationary electromagnetic field

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It is shown that, for a 3×3 correlation matrix $W_{ij}(\mathbf{r}, \mathbf{r}, \omega)$, $(i, j = x, y, z)$ of the electric vector of a random, stationary electromagnetic field to represent a field that is completely polarized at a point $\mathbf{r}$ and frequency ω, each element of the matrix must factorize. More precisely, a necessary and sufficient condition for the correlation matrix to represent a fully polarized field at a point $\mathbf{r}$ is that the matrix has the form $W_{ij}(\mathbf{r}, \mathbf{r}, \omega) = \mathcal{E}_i^*(\mathbf{r}, \omega)\mathcal{E}_j(\mathbf{r}, \omega)$, where $\mathcal{E}_i(\mathbf{r}, \omega)$ $(i = x, y, z)$ are deterministic functions, i.e., that all pairs of the Cartesian components of the electric field at a point $\mathbf{r}$ and frequency ω are completely correlated.

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A problem of considerable current interest, especially in the area of near-field optics, is the determination of the state of polarization of a randomly fluctuating electromagnetic field close to a source or to a scattering body. In these regions the field is not beamlike. The problem is nontrivial, because even the question of whether the concept of the degree of polarization of a general (i.e., not beamlike) random electromagnetic field is meaningful has not as yet been clarified, as is evident from contradictory claims about this question made in the literature (see, for example, Refs. 1–3). To clarify this problem it is necessary to know how to represent a fully polarized random statistically stationary electromagnetic field. The purpose of this Letter is to answer this particular question. In a subsequent publication, we will show that the elucidation of this question is crucial to the understanding of whether, and in what sense, the concept of a degree of polarization of an arbitrary, statistically stationary electromagnetic field is meaningful.

Let us begin with some remarks about polarization properties of random electromagnetic beams that propagate close to the z direction. The electric vector of a beamlike field is said to be completely polarized at a point $\mathbf{r}$ if the end point of the electric vector at that point moves with increasing time on an ellipse (which of course may degenerate into a straight line or a circle in special cases). It is worth noting that the field does not have to be monochromatic to be fully polarized. All that is necessary is that the ratio E_y/E_x of two mutually orthogonal components of the electric vector perpendicular to the direction of propagation is constant (see, for example, Ref. 4, Sect. 6.3.2).

Expressed in a different though equivalent way, the electric vector of a random statistically stationary electromagnetic beam propagating close to the z direction is completely polarized at a point $\mathbf{r}$, and frequency ω, if its spectral degree of correlation,

$$\mu_{xy}(\mathbf{r}, \omega) = \frac{\langle E_x^*(\mathbf{r}, \omega)E_y(\mathbf{r}, \omega)\rangle}{[\langle E_x^*(\mathbf{r}, \omega)E_x(\mathbf{r}, \omega)\rangle]^{1/2}\,[\langle E_y^*(\mathbf{r}, \omega)E_y(\mathbf{r}, \omega)\rangle]^{1/2}}, \quad (1)$$

is unimodular, i.e., if $|\mu_{xy}(\mathbf{r}, \omega)| = 1$. In Eq. (1), $E_x(\mathbf{r}, \omega)$ and $E_y(\mathbf{r}, \omega)$ are members of a statistical ensemble of space-dependent parts of monochromatic realizations $E_x(\mathbf{r}, \omega)\exp(-i\omega t)$, $E_y(\mathbf{r}, \omega)\exp(-i\omega t)$ that represent the components of the random complex electric field at a point $\mathbf{r}$ and at frequency ω, and the angle brackets denote the average, taken over an ensemble of realizations of the electric field (see Ref. 4, Sect. 4.7.1). The constraint $|\mu_{xy}(\mathbf{r}, \omega)| = 1$ implies that the components $E_x(\mathbf{r}, \omega)$ and $E_y(\mathbf{r}, \omega)$ are completely correlated.

Let us now turn our attention to a random electromagnetic field that is not beamlike. It is known that any monochromatic field, whether or not it is beamlike, is necessarily polarized at each point, but the polarization ellipses may have a different shape, and their planes may have different orientations at different points (Ref. 5, Sect. 1.4.3). Actually, as already noted, the field at some point $\mathbf{r}$ need not be monochromatic to be fully polarized at that point. In the space–frequency representation that we are now using, this can be seen from the following argument.[6]

Let us consider a statistical ensemble $[E_i(\mathbf{r}, \omega)]$ $(i = x, y, z)$ (see Ref. 4, Sect. 4.7.2) of the electric field at point $\mathbf{r}$ and frequency ω. Suppose that

$$E_i(\mathbf{r}, \omega) = e_i(\mathbf{r}, \omega) U(\mathbf{r}, \omega)\,, \tag{2}$$

where $e_i(\mathbf{r}, \omega)$ are deterministic functions and $U(\mathbf{r}, \omega)$ is a random function. In other words, e_i is the same for each member of the ensemble but U differs from member to member of the E_i ensemble.

It follows at once from Eq. (2) that the ratio

$$\frac{E_i(\mathbf{r}, \omega)}{E_j(\mathbf{r}, \omega)} = \frac{e_i(\mathbf{r}, \omega)}{e_i(\mathbf{r}, \omega)} \tag{3}$$

is deterministic. Two fields for which the ratios E_i/E_j $(i = x, y, z)$ have the same (deterministic) value may be said to be statistically similar. Using this deterministic relationship between the vector components, we may define an equivalent electromagnetic field

$$\mathscr{E}_i(\mathbf{r}, \omega)\exp(i\omega t) = e_i(\mathbf{r}, \omega)[\langle |U(\mathbf{r}, \omega)|^2\rangle]^{1/2}\exp(-i\omega t)\,, \tag{4}$$

that, just as a monochromatic field, will necessarily be fully polarized at each point; i.e., the end point of the electric field will move on an ellipse.

Let us now consider any statistically stationary field. In general such a field will, of course, not be fully polarized. We may characterize the second-order correlation properties of its fluctuating electric field at points $\mathbf{r}_1$ and $\mathbf{r}_2$ and frequency ω by a 3×3 cross-spectral density matrix (see Ref. 4, Sect. 6.6.1):

$$W_{ij}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle E_i^*(\mathbf{r}_1, \omega) E_j(\mathbf{r}_2, \omega)\rangle\,, \tag{5}$$

where the subscripts i and j label the Cartesian components of the (generally complex) electric field. Since we are interested in the state of polarization of the field at some particular point represented by a position vector $\mathbf{r}$ and frequency ω, we need only consider the matrix (5) for $\mathbf{r}_1 = \mathbf{r}_2 \equiv \mathbf{r}$.

We will now establish the following theorem: *A necessary and sufficient condition for the electric vector of a statistically stationary electromagnetic field to be completely polarized at a point* $\mathbf{r}$ *is that each element of the matrix* $W_{ij}(\mathbf{r}, \mathbf{r}, \omega)$ *factorizes in the form*

$$W_{ij}(\mathbf{r}, \mathbf{r}, \omega) = \mathscr{E}_i^*(\mathbf{r}, \omega)\mathscr{E}_j(\mathbf{r}, \omega)\,. \tag{6}$$

This factorization implies that the Cartesian components $E_i(\mathbf{r}, \omega)$, $E_j(\mathbf{r}, \omega)$ are completely correlated for all $i, j = x, y, z$. To prove that such factorization is a necessary condition for the field to be completely polarized, we start from the ensemble representation of a fully polarized field. In view of Eq. (4), the cross-spectral density matrix of such a field is

$$\begin{aligned} W_{ij}(\mathbf{r}, \mathbf{r}, \omega) &\equiv \langle E_i^*(\mathbf{r}_1, \omega)E_j(\mathbf{r}_2, \omega)\rangle \\ &= \langle {e_i}^*(\mathbf{r}, \omega)U^*(\mathbf{r}, \omega)e_j(\mathbf{r}, \omega)U(\mathbf{r}, \omega)\rangle \\ &= \langle {e_i}^*(\mathbf{r}, \omega)e_j(\mathbf{r}, \omega)\langle U^*(\mathbf{r}, \omega)U(\mathbf{r}, \omega)\rangle \\ &= \mathscr{E}_i^*(\mathbf{r}, \omega)\mathscr{E}_j(\mathbf{r}, \omega)\,, \end{aligned} \tag{7}$$

say. Evidently such a factorization of all the elements of the correlation matrix expresses a necessary condition for complete polarization of the electric field at a point $\mathbf{r}$ and frequency ω.

To demonstrate that the factorization is also a sufficient condition we note that because the matrix $W_{ij}(\mathbf{r}, \mathbf{r}, \omega)$ is a nonnegative definite Hermitian matrix (Ref. 4, Sect. 6.6.1), it can be diagonalized by a unitary transformation and, moreover, its eigenvalues, λ, are necessarily real and nonnegative. The eigenvalues are solutions of the equation

$$\det[\overleftrightarrow{W}(\mathbf{r}, \mathbf{r}, \omega) - \lambda I] = 0\,, \tag{8}$$

where det denotes the determinant and I is the unit matrix. On substituting from Eq. (6) into Eq. (8) we find that

$$\begin{aligned} &(|\mathscr{E}_x|^2 - \lambda)[(|\mathscr{E}_y|^2 - \lambda)(|\mathscr{E}_z|^2 - \lambda) - |\mathscr{E}_y|^2|\mathscr{E}_z|^2] \\ &\quad - \mathscr{E}_x^*\mathscr{E}_y[\mathscr{E}_y^*\mathscr{E}_x(|\mathscr{E}_z|^2 - \lambda) - |\mathscr{E}_z|^2\mathscr{E}_y^*\mathscr{E}_x] \\ &\quad + \mathscr{E}_x^*\mathscr{E}_z[|\mathscr{E}_y|^2\mathscr{E}_z^*\mathscr{E}_x - \mathscr{E}_z^*\mathscr{E}_x(|\mathscr{E}_y|^2 - \lambda)] = 0\,, \end{aligned} \tag{9}$$

where of course the arguments of $\mathscr{E}_x$, $\mathscr{E}_y$, and $\mathscr{E}_z$ are functions of $\mathbf{r}$ and ω.

After lengthy but straightforward calculations, Eq. (9) can be shown to imply that

$$\lambda^2[\lambda - \mathrm{Tr}\,\overleftrightarrow{W}(\mathbf{r}, \mathbf{r}\omega)] = 0\,, \tag{10}$$

where Tr denotes the trace. Equation (10) shows that the matrix $\overleftrightarrow{W}(\mathbf{r}, \mathbf{r}, \omega)$ has only one nonzero eigenvalue that is proportional to the average electric energy density of the field at point $\mathbf{r}$ because

$$\begin{aligned} \lambda_1 = \mathrm{Tr}\,\overleftrightarrow{W}(\mathbf{r}, \mathbf{r}, \omega) &= \langle E_x^*(\mathbf{r}, \omega)E_x(\mathbf{r}, \omega)\rangle \\ &\quad + \langle E_y^*(\mathbf{r}, \omega)E_y(\mathbf{r}, \omega)\rangle + \langle E_z^*(\mathbf{r}, \omega)E_z(\mathbf{r}, \omega)\rangle\,. \end{aligned} \tag{11}$$

To explicitly determine the state of polarization represented by a cross-spectral density matrix each element of which factorizes, it is necessary to transform (rotate) the coordinate system appropriately. We now perform this rotation.

In general, the electric field at point $\mathbf{r}$ will be elliptically polarized, and the eigenvector v_1 corresponding to the nonzero eigenvalue λ_1 is a complex vector. Just as the eigenvalue is associated with the electric energy density of the field at the point, the eigenvector v_1 can be associated with the equivalent electric field given by Eq. (4). Up to a constant, $v_1 \equiv [\mathscr{E}_x^*, \mathscr{E}_y^*, \mathscr{E}_z^*]^T$, where T denotes the transpose, the vector is just the complex conjugate of the equivalent electric field at point $\mathbf{r}$. A treatment similar to that given in Sect. 1.4.2 of Ref. 5 gives the polarization ellipse of the electric field at point $\mathbf{r}$. We provide an equivalent argument through real rotations defined in terms of the component of the eigenvector v_1 (equivalently, in terms of the electric field components). This rotation is given by a matrix of the form

$$R = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & -\cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{12}$$

where the angles of rotation are

$$\theta(\mathbf{r}, \omega) = \arctan\left[\frac{\mathcal{E}_z(\mathbf{r}, \omega)}{\mathcal{E}_x(\mathbf{r}, \omega)\cos(\alpha) + \mathcal{E}_y(\mathbf{r}, \omega)\sin(\alpha)}\right], \tag{13a}$$

$$\alpha(\mathbf{r}, \omega) = \arctan\left\{\frac{|\mathcal{E}_x(\mathbf{r}, \omega)| \sin[\phi_z(\mathbf{r}, \omega) - \phi_x(\mathbf{r}, \omega)]}{|\mathcal{E}_y(\mathbf{r}, \omega)| \sin[\phi_y(\mathbf{r}, \omega) - \phi_z(\mathbf{r}, \omega)]}\right\}. \tag{13b}$$

The cross-spectral density matrix in the new coordinate system is

$$\tilde{\overleftrightarrow{W}}(\mathbf{r}, \mathbf{r}, \omega) = R\overleftrightarrow{W}(\mathbf{r}, \mathbf{r}, \omega)R^{-1} = \begin{bmatrix} \mathcal{E}^*_{x'}(\mathbf{r}, \omega) & \mathcal{E}_{x'}(\mathbf{r}, \omega) & \mathcal{E}^*_{x'}(\mathbf{r}, \omega) & \mathcal{E}_{y'}(\mathbf{r}, \omega) & \mathcal{E}^*_{x'}(\mathbf{r}, \omega) & \mathcal{E}_{z'}(\mathbf{r}, \omega) \\ \mathcal{E}^*_{y'}(\mathbf{r}, \omega) & \mathcal{E}_{x'}(\mathbf{r}, \omega) & \mathcal{E}^*_{y'}(\mathbf{r}, \omega) & \mathcal{E}_{y'}(\mathbf{r}, \omega) & \mathcal{E}^*_{y}(\mathbf{r}, \omega) & \mathcal{E}_{z'}(\mathbf{r}, \omega) \\ \mathcal{E}^*_{z'}(\mathbf{r}, \omega) & \mathcal{E}_{x'}(\mathbf{r}, \omega) & \mathcal{E}^*_{z'}(\mathbf{r}, \omega) & \mathcal{E}_{y'}(\mathbf{r}, \omega) & \mathcal{E}^*_{z}(\mathbf{r}, \omega) & \mathcal{E}_{z'}(\mathbf{r}, \omega) \end{bmatrix}, \tag{14}$$

where

$$\mathcal{E}_{x'}(\mathbf{r}, \omega) = \mathcal{E}_x(\mathbf{r}, \omega)\cos(\theta)\cos(\alpha) + \mathcal{E}_y(\mathbf{r}, \omega)\cos(\theta)\sin(\alpha) - \mathcal{E}_z(\mathbf{r}, \omega)\sin(\theta), \tag{15a}$$

$$\mathcal{E}_{y'}(\mathbf{r}, \omega) = -\mathcal{E}_x(\mathbf{r}, \omega)\sin(\alpha) + \mathcal{E}_y(\mathbf{r}, \omega)\cos(\alpha), \tag{15b}$$

$$\mathcal{E}_{z'}(\mathbf{r}, \omega) = -\mathcal{E}_x(\mathbf{r}, \omega)\sin(\theta)\cos(\alpha) - \mathcal{E}_y(\mathbf{r}, \omega)\sin(\theta)\sin(\alpha) + \mathcal{E}_z(\mathbf{r}, \omega)\cos(\theta). \tag{15c}$$

On substituting for θ and α from Eq. (13) into Eq. (15), one finds that $\mathcal{E}_{z'}(\mathbf{r}, \omega) = 0$, which implies that the electric field is confined to the x', y' plane. It should be noted that, while the ratio $\mathcal{E}_z(\mathbf{r}, \omega)/[\mathcal{E}_x(\mathbf{r}, \omega)\cos(\alpha) + \mathcal{E}_y(\mathbf{r}, \omega)\sin(\alpha)]$ appears to be a complex quantity, its imaginary component is identically zero, and hence both θ and α are real angles. We have thus reduced the matrix representation of the three-dimensional field at a point $\mathbf{r}$ to that of a locally two-dimensional field represented by the matrix

$$\tilde{\overleftrightarrow{W}}(\mathbf{r}, \mathbf{r}, \omega) = \begin{bmatrix} \mathcal{E}^*_{x'}(\mathbf{r}, \omega)\mathcal{E}_{x'}(\mathbf{r}, \omega) & \mathcal{E}^*_{x'}(\mathbf{r}, \omega)\mathcal{E}_{y'}(\mathbf{r}, \omega) & 0 \\ \mathcal{E}^*_{y'}(\mathbf{r}, \omega)\mathcal{E}_{x'}(\mathbf{r}, \omega) & \mathcal{E}^*_{y'}(\mathbf{r}, \omega)\mathcal{E}_{y'}(\mathbf{r}, \omega) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{16}$$

The 2×2 submatrix in Eq. (16), with the factorized terms, will be recognized as a correlation matrix of a completely polarized field confined to the x', y' plane through point $\mathbf{r}$ (Ref. 4, Sect. 6.3.2). We have thus demonstrated the sufficiency condition, showing that factorization of all elements implies a fully polarized field. In the degenerate case when the field is linearly polarized, the three components of the eigenvector v_1 will have the same phase. In this case angle α takes on the value $\alpha = \arctan(|\mathcal{E}_x|/|\mathcal{E}_y|)$, which is just the limiting case of α with $\phi_z - \phi_x = \phi_y - \phi_z$.

Thus, we have proved that a necessary and sufficient condition for an electric cross-spectral density matrix to represent a completely polarized field at a point $\mathbf{r}$ and frequency ω is that each element of the matrix at $\mathbf{r}_1 = \mathbf{r}_2 \equiv \mathbf{r}$ factorizes in the form $W_{ij}(\mathbf{r}, \mathbf{r}, \omega) = \mathcal{E}^*_i(\mathbf{r}, \omega)\mathcal{E}_j(\mathbf{r}, \omega)$.

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6. A referee of this paper pointed out that, in Ref. 1, a definition of the degree of polarization of three-dimensional fields was put forward that gives the value unity in this limiting case. This is, however, fortuitous, because the definition proposed in Ref. 1 is purely formal and does not have the meaning of the ratio of the intensity of a fully polarized field to the total intensity at a point.