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Degree of polarization of statistically stationary electromagnetic fields

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Abstract

The analysis presented in this paper resolves an outstanding controversial issue of statistical optics, concerning the existence of the degree of polarization of any random, statistically stationary electromagnetic field. We show that the second-order electric spectral correlation matrix at any point in such a field may be uniquely expressed as the sum of three matrices, the first of which represents a completely polarized contribution. The ratio of the average intensity of the polarized part to the total average intensity provides a unique and unambiguous definition of the spectral degree of polarization of the electric field. It may be expressed by a simple formula in terms of the eigenvalues of the correlation matrix of the electric field and it reduces, for the two-dimensional case, to the usual well-known expression for the degree of polarization of beam-like fields. The results of this paper are of special interest for near-field optics.

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An important outstanding unsolved problem in the statistical theory of electromagnetic radiation concerns the question of whether, and if so how, one can characterize its state of polarization. In the special case when the field has a local structure of a plane wave, more precisely when it is "beam-like" (also referred to as a paraxial field or a two-dimensional field), one may characterize its second-order spectral correlation properties at any point in space by a 2×2 correlation matrix, usually called a coherency matrix or a polarization matrix. It may be uniquely expressed as the sum of two matrices, one of which represents a fully polarized part and the other a completely

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unpolarized part. The ratio of the average intensity (electric energy density) of the polarized part to the total averaged intensity is identified with the degree of polarization of the field at that point. It may be represented in terms of the Stokes parameters, which are linear combinations of the elements of the 2×2 coherency matrix and which turn out to be proportional to the coefficients in a decomposition of that matrix onto Pauli spin matrices [1, p. 349]. This theory is not only the basis of the whole of polarization optics in classical physics but has an important counterpart in the quantum treatment of polarization [2].

The question as to whether the concept of the degree of polarization can be generalized from beam-like fields to three-dimensional random electromagnetic fields has been considered for many years but no satisfactory solution has been found up to now. In fact there are several contradictory claims made in the literature about this subject. For example in a relatively recent modern text [3, p. 121] it is stated that "For $N = 2$ it was possible to consider an arbitrary (partially polarized) state as an incoherent mixture of a pure state and the completely unpolarized state. However, for $N = 3$, one cannot generally express an arbitrary state as a mixture of a fully polarized and a completely unpolarized part". On the other hand, in a recent paper [4], a definition of a degree of polarization of a three-dimensional field was introduced, though with some hesitation, and without any discussion of its apparent physical meaning. The decomposition of the electric correlation matrix used in that paper and in some earlier treatments of the problem makes use of the Gell–Mann matrices in place of the Pauli's spin matrices of the traditional ("two-dimensional") theory of polarization. References to numerous other publications making contradictory claims are given in [3,4].

A clear and unambiguous solution to this problem is of considerable current interest in the rapidly expanding area of near-field optics. The electromagnetic field close to a radiating source or to a scattering object is not beam-like and its polarization properties cannot, therefore, be analyzed within the framework of the traditional polarization optics. Consequently, a more general treatment is needed.

In the present paper we address this problem. Specifically we investigate whether for a random, statistically stationary electromagnetic field it is possible to introduce, at each point in space, a unique degree of polarization. To answer this question we first show that the 3×3 cross-spectral density matrix of the electric field at any point in space may be uniquely expressed as the sum of three matrices. The first represents a fully polarized field, while the other two matrices are related to unpolarized contributions. The ratio of the averaged intensity of the polarized part to the averaged intensity of the total field at that point may evidently be identified with the long-sought definition of the degree of polarization at any chosen point in a random, statistically stationary, three-dimensional electromagnetic field. It is found to be expressible in a simple way in terms of the eigenvalues of the 3×3 cross-spectral density matrix of the electric field and reduces to the usual expression for the degree of polarization when the field is beam-like.

To derive these results, we first recall that the second-order correlation properties of such a field may be characterized by the 3×3 correlation matrix

$$\overleftrightarrow{W}(\mathbf{r}_1, \mathbf{r}_2, \omega) = [W_{ij}(\mathbf{r}_1, \mathbf{r}_2, \omega)] = [\langle E_i^*(\mathbf{r}_1, \omega) E_j(\mathbf{r}_2, \omega) \rangle]. \tag{1}$$

Here, E_i $(i = x, y, z)$ is a member of a statistical ensemble of monochromatic realizations of the fluctuating electric field at a point $\mathbf{r}$ and frequency ω and the angular brackets denote the average, taken over an ensemble of realizations in the sense of the coherence theory in the space-frequency domain [1, Section 4.7.1]. We will now show that this matrix evaluated at a point $\mathbf{r}_1 = \mathbf{r}_2 \equiv \mathbf{r}$, may be uniquely represented as the sum of contributions of three fields which are mutually uncorrelated. Unlike the rather formal decomposition of three-dimensional fields onto the Gell–Mann matrices, which has frequently been used in the past in attempts to analyze polarization properties of such fields (see, for example [3, Section 3.1.6.7] and [4]), the new decomposition has a useful and physically clear interpretation.

Since the cross-spectral density matrix at a single point $\mathbf{r}$ and frequency ω is a non-negative definite Hermitian matrix, there is a unitary matrix

$$\overleftrightarrow{U} = \begin{bmatrix} E^*_{x1} & E^*_{x2} & E^*_{x3} \\ E^*_{y1} & E^*_{y2} & E^*_{y3} \\ E^*_{z1} & E^*_{z2} & E^*_{z3} \end{bmatrix} \quad (2)$$

that will diagonalize $\overleftrightarrow{W}$. The elements of the diagonal matrix are the eigenvalues of $\overleftrightarrow{W}$, which are necessarily real and non-negative. This unitary matrix can be represented in a unique form by ordering the eigenvalues as $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0$.

The diagonalization of the cross-spectral density matrix then gives

$$\overleftrightarrow{U}^{\dagger} \overleftrightarrow{W} \overleftrightarrow{U} = \overleftrightarrow{D} = \sum_{1} \lambda_1 \overleftrightarrow{D}_i = \lambda_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3)$$

Defining a set of matrices $[w_i]$ $(i = 1, 2, 3)$ corresponding to the $\overleftrightarrow{D}_i$ in the original coordinate frame, one readily finds that

$$\overleftrightarrow{w}(\mathbf{r}, \mathbf{r}, \omega) \equiv \overleftrightarrow{U} \overleftrightarrow{D}_i \overleftrightarrow{U}^{\dagger} = \begin{bmatrix} |E_{xi}|^2 & E_{xi}E^*_{yi} & E_{xi}E^*_{xi} \\ E^*_{xi}E_{yi} & |E_{yi}|^2 & E_{yi}E^*_{xi} \\ E^*_{xi}E_{xi} & E^*_{yi}E_{xi} & |E_{xi}|^2 \end{bmatrix}^{\dagger}, \quad (4)$$

which represents a fully polarized field [5] whose average intensity at a point $\mathbf{r}$ and at frequency ω, is independent of the subscript i. By fully polarized, we mean a field such that the end point of the electric field vector moves on an ellipse or, equivalently, a field for which the cross-spectral density matrix evaluated at a point $\mathbf{r}$ has a single non-zero eigenvalue [5]. It is evident that Eq. (3) expresses the matrix $\overleftrightarrow{W}(\mathbf{r}, \mathbf{r}, \omega)$ uniquely as the sum of three uncorrelated, orthogonally polarized fields. This interpretation of Eq. (3) suggests the following decomposition of the 3×3 matrix $\overleftrightarrow{W}(\mathbf{r}, \mathbf{r}, \omega)$ which is analogous to the decomposition of the 2×2 correlation matrix of a beam-like field:

$$\overleftrightarrow{W}(\mathbf{r}, \mathbf{r}, \omega) \equiv \overleftrightarrow{M}^{(1)} + \overleftrightarrow{M}^{(2)} + \overleftrightarrow{M}^{(3)}, \quad (5)$$

where

$$\overleftrightarrow{M}^{(1)}(\mathbf{r}, \mathbf{r}, \omega) \equiv (\lambda_1 - \lambda_2) \overleftrightarrow{U} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^{\dagger} = (\lambda_1 - \lambda_2)\overleftrightarrow{w}_1, \quad (6a)$$

$$\overleftrightarrow{M}^{(2)}(\mathbf{r}, \mathbf{r}, \omega) \equiv (\lambda_2 - \lambda_3) \overleftrightarrow{U} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^{\dagger} = (\lambda_2 - \lambda_3)(\overleftrightarrow{w}_1 + \overleftrightarrow{w}_2), \quad (6b)$$

$$\overleftrightarrow{M}^{(3)}(\mathbf{r}, \mathbf{r}, \omega) \equiv \lambda \overleftrightarrow{U} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{\dagger} = \lambda_3(\overleftrightarrow{w}_1 + \overleftrightarrow{w}_2 + \overleftrightarrow{w}_3). \quad (6c)$$

The matrix $\overleftrightarrow{M}^{(1)}(\mathbf{r}, \mathbf{r}, \omega)$ represents a field which is completely polarized at a point $\mathbf{r}$ at frequency ω, as was shown in a recent paper [5]. The matrices $\overleftrightarrow{M}^{(2)}(\mathbf{r}, \mathbf{r}, \omega)$ and $\overleftrightarrow{M}^{(3)}(\mathbf{r}, \mathbf{r}, \omega)$, being sums of two- and three mutually orthogonally polarized fields represent contributions from unpolarized two- and three-dimensional components. This interpretation will be discussed in another publication [6].

It is clear from these results that one can associate with any point in a random, statistically stationary electromagnetic field a unique spectral degree of polarization, which is the ratio of the averaged intensity $I^{(p)}$ of the polarized part of the field to the total averaged intensity I of the field at that point, i.e.,

$$P \equiv \frac{I^{(p)}}{I} = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}. \quad (7)$$

It is clear that $0 \leqslant P \leqslant 1$ with $P = 0$ representing completely unpolarized field and $P = 1$ completely polarized field at the point $\mathbf{r}$. It is to be noted that when the field is two-dimensional, $\lambda_3 = 0$, and so Eq. (7) then reduces to the usual expression for the degree of polarization [1, Eq. (6.3–31)].

We will illustrate our main results by an example. Suppose that a random evanescent wave with electric field represented by a statistically stationary ensemble whose realizations are of the form

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{e}_0 x_i e^{i\mathbf{k}\cdot\mathbf{r}} \quad (\mathbf{x} = x_1, x_2, x_3,\ \mathbf{x}^2 = 1, x_k = 0), \tag{8}$$

where $\mathbf{e}_0$ is a random variable, is incident on a half-space $z \geqslant 0$ containing blackbody radiation. We assume that the wave is linearly polarized along the x-direction and decays exponentially in the positive z-direction, $\mathbf{e}_0$ is a (generally complex) amplitude and the complex wave vector $\mathbf{k}$ has components $(0, k_y, k_z = k_z' + ik_z'')$ with k_z' and k_z'' real, $\mathbf{k}\cdot\mathbf{k}^* = \omega^2/c^2$ and asterisk denoting the complete conjugate. The correlation matrix of this ensemble has the form

$$\begin{aligned} W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\omega) =& \langle |e_0|^2 x_i x_j \exp[-i(\mathbf{k}^*\cdot\mathbf{r}_1 - \mathbf{k}\cdot\mathbf{r}_2)]\rangle \\ =& \langle |e_0|^2\rangle x_i x_j [-ik_y(y_1 - y_2) \\ &- ik_z'(z_1 - z_2) - k_z''(z_1 + z_2)]. \end{aligned} \tag{9}$$

With $\mathbf{r}_2 = \mathbf{r}_1$ we evidently have

$$W_{ij}^{(e)}(\mathbf{r},\mathbf{r},\omega) = \langle |e_0|^2\rangle x_i x_j \exp(-2k_z'' z). \tag{10}$$

The cross-spectral density matrix of blackbody radiation is given by (see [7, Eq. (3.10)])

$$\begin{aligned} W_{ij}^{(bb)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \pi A\{&\delta_{ij}[j_0(R) - (1/R)j_1(R)] \\ &+ (R_iR_j/R^2)j_2(R)\}, \end{aligned} \tag{11}$$

where $R = (\omega/c)|\mathbf{r}_2 - \mathbf{r}_1|$ and

$$A = \frac{h}{\pi^2}\left(\frac{\omega}{c}\right)^3 \frac{1}{\exp\{h\omega/2\pi KT\} - 1} \tag{12}$$

is the Plank distribution law, the j's are spherical Bessel functions and δ_{ij} is the Kronecker delta symbol. Taking the limit as $|\mathbf{r}_2 - \mathbf{r}_1| \to 0$, we readily find that

$$W_{ij}^{(bb)}(\mathbf{r},\mathbf{r},\omega) = \frac{2\pi A}{3}\delta_{ij}. \tag{13}$$

For the total field we have

$$W_{ij}(\mathbf{r},\mathbf{r},\omega) = W_{ij}^{(e)}(\mathbf{r},\mathbf{r},\omega) + W_{ij}^{(bb)}(\mathbf{r},\mathbf{r},\omega) \tag{14}$$

or, using Eqs. (10) and (13),

$$W_{ij}(\mathbf{r},\mathbf{r},\omega) = I^{(e)} x_i x_j \exp(-2k_z'' z) + I^{(bb)}\delta_{ij}, \tag{15}$$

where $I^{(e)}$ is the initial intensity of the evanescent field, i.e. the intensity at $z = 0$ and $I^{(bb)}$ is the intensity of the blackbody field. The eigenvectors of the matrix (15) are

$$u_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\quad u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\quad u_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{16}$$

and the eigenvalues are

$$\begin{aligned} &\lambda_1 = I^{(bb)} + I^{(e)}\exp(-2k_z'' z), \\ &\lambda_2 = \lambda_3 = I^{(bb)}. \end{aligned} \tag{17}$$

The degree of polarization of the total field is obtained by substituting from Eq. (17) into our general formula (7) and one readily finds that it is given by the expression

$$P(z) = \frac{1}{3(I^{(bb)}/I^{(e)})\exp(2k_z'' z) + 1}. \tag{18}$$

In Fig. 1 the degree of polarization of the total field is plotted as a function of $k_z'' z$ for some selected values of the ratio $I^{(bb)}/I^{(e)}$. It shows that the degree of polarization $P(z)$ decreases with the distance z from the boundary and that $P(z) \to 0$ as $I^{(bb)}/I^{(e)} \to \infty$ and $P(z) \to 1$ as $I^{(bb)}/I^{(e)} \to 0$, as one would expect.

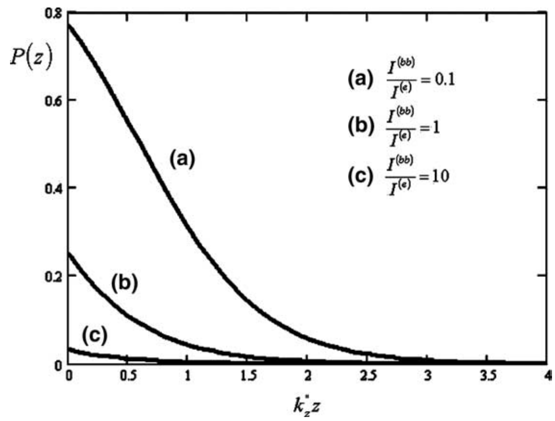

Fig. 1. The degree of polarization as function of z of a field consisting of a superposition of an ensemble of evanescent waves of random amplitude and of blackbody radiation occupying the half-space $z > 0$. The evanescent wave decays exponentially in amplitude with increasing z. $I^{(bb)}$ denotes the intensity of the blackbody field $I^{(e)}$ denotes the intensity of the evanescent wave at the plane $z = 0$.

To summarize, we have resolved an outstanding controversial issue of statistical optics by introducing a definition for the degree of polarization of a general (i.e. not necessarily beam-like) statistically stationary three-dimensional electromagnetic field. We have first demonstrated that the electric cross-spectral density matrix of the field at any point **r** can be uniquely represented as a sum of three matrices, which have a clear physical meaning. This led to an unambiguous definition of the degree of polarization at any point **r** of any three-dimensional statistically stationary electromagnetic field, as the ratio of the averaged intensity of the polarized component to the total averaged intensity of the field at that point. For beam-like fields, our definition of the degree of polarization was shown to reduce to the usual one.

The decomposition and the definition of the degree of polarization of any random, statistically stationary electromagnetic field which we have introduced is likely to find uses in many areas of physics ranging from near-field optics to medical and biological physics, where such fields are routinely encountered.

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