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Physical significance of complete spatial coherence of optical fields

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Abstract

We show that complete coherence of light fluctuations at two points in a statistically stationary optical field implies that the fluctuations are either identical or are proportional to each other, a property which may be called statistical similarity. In particular for light to be completely coherent it need not be monochromatic nor do the fluctuations need to be deterministic.

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It is frequently but nevertheless incorrectly asserted that in order that an optical field is spatially coherent, it has to be monochromatic. By spatially coherent light vibrations at points $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$ we mean, as was first clarified by Zernike in a classic paper ([1], see also [2]) that if light from these two points is superposed, it will form interference fringes with maximum possible visibility, namely unity. In mathematical terms this means, that for some value τ_0 of the time delay between the two beams which propagate from the points P_1 and P_2 to the observation plane, the degree of coherence [3]

$$\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)}{\sqrt{\Gamma(\mathbf{r}_1, \mathbf{r}_1, 0)}\sqrt{\Gamma(\mathbf{r}_2, \mathbf{r}_2, 0)}} \tag{1a}$$

$$= \frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)}{\sqrt{I(\mathbf{r}_1)}\sqrt{I(\mathbf{r}_2)}} \tag{1b}$$

is unimodular. In this formula $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is the mutual coherence function, defined by the formula

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle V^*(\mathbf{r}_1, t)V(\mathbf{r}_2, t+\tau)\rangle, \tag{2a}$$

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and $I(\mathbf{r}_i)$ $(i = 1, 2)$ is the average intensity at $\mathbf{r}_i$ defined by the formula

$$I(\mathbf{r}_i) \equiv \Gamma(\mathbf{r}_i, \mathbf{r}_i, 0) = \langle V^*(\mathbf{r}_i, t)V(\mathbf{r}_i, t)\rangle, \tag{2b}$$

where $V(\mathbf{r}, t)$ represents a fluctuating field at the point $\mathbf{r}$ and time t, asterisks denote the complex conjugate and the angular brackets denote the expectation value taken over the ensemble of the field, assumed to be stationary. If the ensemble is also ergodic as is usually the case, the expectation value may be replaced by the time average.

In order to elucidate the physical significance of complete coherence between field fluctuations at the two points we introduce the concept of *statistical similarity*. We say that the fluctuating field $V(\mathbf{R}, t)$ at two points $P_1(\mathbf{R}_1)$ and $P_2(\mathbf{R}_2)$ is statistically similar if, for some value $\tau = \tau_0$,

$$V(\mathbf{R}_2, t + \tau_0) = \beta(\mathbf{R}_1, \mathbf{R}_2, \tau_0)V(\mathbf{R}_1, t), \tag{3}$$

where $\beta(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$ is a deterministic function.

We will prove the following theorem.

Theorem. *A fluctuating optical field is spatially fully coherent at a given pair of points $\mathbf{R}_1$ and $\mathbf{R}_2$ if and only if the field fluctuations at these points are statistically similar in the sense of Eq.* (3) *with*

$$\beta(\mathbf{R}_1, \mathbf{R}_2, \tau_0) = \sqrt{\frac{I(\mathbf{R}_2)}{I(\mathbf{R}_1)}}\mathbf{e}^{-i\theta(\mathbf{R}_1,\mathbf{R}_2,\tau_0)},$$

where τ_0 is some constant value of τ, $\theta(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$ is the phase of the (*unimodular degree of coherence*) *$\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$ and $I(\mathbf{R}_i) = \langle V^*(\mathbf{R}_i, t)V(\mathbf{R}_i, t)\rangle$ denotes the average intensity at the point $\mathbf{R}_i$ $(i = 1, 2)$.*

Proof. We will first prove that statistical similarity is a *sufficient* condition for $|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)|$ to equal to unity. On substituting from Eq. (3) into Eq. (2), with $\mathbf{r}_1 = \mathbf{R}_1, \mathbf{r}_2 = \mathbf{R}_2$ and $\tau = \tau_0$, we have

$$\begin{aligned}\Gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0) &= \langle V^*(\mathbf{R}_1, t)\beta(\mathbf{R}_1, \mathbf{R}_2, \tau_0)V(\mathbf{R}_1, t)\rangle \\ &= \beta(\mathbf{R}_1, \mathbf{R}_2, \tau_0)I(\mathbf{R}_1).\end{aligned} \tag{4}$$

On substituting from Eq. (4) into the expression (1), with the choice $\mathbf{r}_1 = \mathbf{R}_1, \mathbf{r}_2 = \mathbf{R}_2, \tau = \tau_0$ we obtain for the degree of coherence the expression

$$\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0) = \beta(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\sqrt{\frac{I(\mathbf{R}_1)}{I(\mathbf{R}_2)}}. \tag{5}$$

Hence $|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)| = 1$, implies that

$$\left|\beta(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right| = \sqrt{\frac{I(\mathbf{R}_2)}{I(\mathbf{R}_1)}}. \tag{6}$$

Also, we can see at once from Eq. (5) that the phase of $\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$ is equal to the phase of $\beta(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$. We have thus proven the sufficiency condition for the theorem.

To prove that statistically similarity is a *necessary* condition for complete coherence we start with the obvious inequality

$$\left\langle\left|V(\mathbf{R}_2, t + \tau_0) - \alpha(\mathbf{R}_1, \mathbf{R}_2, \tau_0)V(\mathbf{R}_1, t)\right|^2\right\rangle \geqslant 0, \tag{7}$$

where

$$\alpha(\mathbf{R}_1, \mathbf{R}_2, \tau_0) = \left|\alpha(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right|e^{-i\phi(\mathbf{R}_1,\mathbf{R}_2,\tau_0)} \tag{8}$$

is a deterministic function. Written more explicitly, the inequality (7) implies that

$$\begin{aligned}&I(\mathbf{R}_2) + \left|\alpha^*(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right|^2 I(\mathbf{R}_1) - \left|\alpha(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right| \\ &\quad\times \left[\Gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)e^{i\phi(\mathbf{R}_1,\mathbf{R}_2,\tau_0)} + \text{c.c.}\right] \geqslant 0.\end{aligned} \tag{9}$$

Using (1), the inequality (9) may be rewritten in the form

$$\begin{aligned}&I(\mathbf{R}_2) + \left|\alpha(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right|^2 I(\mathbf{R}_1) - 2\sqrt{I(\mathbf{R}_1)}\sqrt{I(\mathbf{R}_2)} \\ &\quad\times \operatorname{Re}\left\{\alpha(\mathbf{R}_1, \mathbf{R}_2, \tau)\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right\} \geqslant 0,\end{aligned} \tag{10}$$

where Re{ } stands for the real part. Let

$$\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0) = \left|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right|e^{-i\psi(\mathbf{R}_1,\mathbf{R}_2,\tau_0)} \tag{11}$$

where $\psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$ is the phase of $\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$. Using Eq. (11) along with Eq. (8) in Eq. (10) yields the following inequality:

$$\begin{aligned}&I(\mathbf{R}_2) + \left|\alpha(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right|^2 I(\mathbf{R}_1) \\ &\quad - 2\sqrt{I(\mathbf{R}_1)}\sqrt{I(\mathbf{R}_2)}\left|\alpha(\mathbf{R}_1, \mathbf{R}_2, \tau)\right|\left|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right| \\ &\quad\times \cos\left(\phi(\mathbf{R}_1, \mathbf{R}_2, \tau_0) - \psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right) \geqslant 0.\end{aligned} \tag{12}$$

By a well-known theorem on non-negative definite quadratic forms [4], the inequality (12) holds for any value of $|\alpha|$ provided that

$$\begin{aligned}&\left|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right| \\ &\quad\times \cos\left(\phi(\mathbf{R}_1, \mathbf{R}_2, \tau_0) - \psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right) \leqslant 1.\end{aligned} \tag{13}$$

Eq. (13) holds for all values of $|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)|$, $\phi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$ and $\psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$. In particular, since it holds when $\cos(\phi(\mathbf{R}_1, \mathbf{R}_2, \tau_0) - \psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)) = 1$, it follows that

$$\left|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)\right| \leqslant 1 \tag{14}$$

which is just the usual constraint on the degree of coherence.

The inequality (13) reduces to an equality only when $|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)| = 1$, and $\cos(\phi(\mathbf{R}_1, \mathbf{R}_2, \tau_0) - \psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)) = 1$, i.e., only when the field fluctuations at the points $\mathbf{R}_1$ and $\mathbf{R}_2$ are mutually completely coherent, with $\tau = \tau_0$ and when $\phi(\mathbf{R}_1, \mathbf{R}_2, \tau_0) = \psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0) \pmod{2\pi}$. When the inequality (13) becomes an equality then (7) likewise becomes an equality, i.e., one then has

$$V(\mathbf{R}_2, t + \tau_0) = \alpha(\mathbf{R}_1, \mathbf{R}_2, \tau_0) V(\mathbf{R}_1, t). \tag{15}$$

Hence $|\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)| = 1$ implies that

$$\alpha(\mathbf{R}_1, \mathbf{R}_2, \tau_0) = \sqrt{\frac{I(\mathbf{R}_2)}{I(\mathbf{R}_1)}} e^{-i\psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)}, \tag{16}$$

where $\psi(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$ is the phase of $\gamma(\mathbf{R}_1, \mathbf{R}_2, \tau_0)$. This completes the proof of the necessary condition. □

The theorem formulated in this Letter, namely that complete coherence implies statistical similarity, has been known in a rudimentary way to the distinguished French physicist Verdet already about 140 years ago [5], before the concept of coherence was introduced. In deriving an expression for what today would be called "the area of coherence" of sunlight on the earth surface, Verdet stated (in French) that the diameter d of the region on the earth's surface in which the vibrations of the light are *in unison* (emphasis added) is about $0.5R\bar{\lambda}/\rho$ where R is the distance from the sun to the earth, ρ is the radius of the sun and $\bar{\lambda}$ is the mean wavelength of sunlight. With $\rho/R \approx 0.005$ radians and $\bar{\lambda} = 5.5 \times 10^{-5}$ cm, appropriate to the sun, the diameter $d \approx 0.02$ m. This figure is in agreement with the diameter of the area of coherence of sunlight on the earth surface, calculated from modern coherence theory [6].

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