Do Solitonlike Self-Similar Waves Exist in Nonlinear Optical Media?

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We show analytically that bright and dark spatial self-similar waves can propagate in graded-index amplifiers exhibiting self-focusing or self-defocusing Kerr nonlinearities. The intensity profiles of the novel waves are identical with those of fundamental bright or dark spatial solitons supported by homogeneous passive waveguides with the same type of nonlinearity. Thus, we reveal a previously unnoticed connection between spatial solitons and self-similar waves. We also suggest that the discovered self-similar waves can be used in a promising scheme for the amplification and focusing of spatial solitons in future all-optical networks.

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The sheer complexity of nonlinear physical systems, which are ubiquitous in nature, has motivated researchers to explore hidden symmetries and order within such systems. In many cases, self-similarity of a complex nonlinear system points to the presence of an internal order, and it is a key to gaining physical insight about its evolution [1]. In particular, the self-similar evolution of a nonlinear wave implies that the wave profile remains unchanged and its amplitude and width simply scale with time or propagation distance.

Self-similar behavior of nonlinear waves has always been one of the central themes in many fields of physics, ranging from hydrodynamics and turbulence [1,2] through plasma physics [3] to nuclear physics [1]. Surprisingly, however, it has only lately attracted the attention of nonlinear optics community, and a relatively few optical self-similar phenomena have been investigated to date [4–13]. In particular, exact self-similar solitary waves have been found in optical fibers whose dispersion, nonlinearity and gain profile are allowed to change with the propagation distance, but the functional forms of these parameters cannot be chosen independently [11,12]. Such self-similar solitary waves were shown to have many features similar to ideal solitons [11]. More recently, these results were generalized to obtain exact matter-wave solitons in harmonically trapped Bose-Einstein condensates with the interaction strength and the trapping potential changing with time in a prescribed fashion [13].

In this Letter, we present *exact analytical* solutions describing spatial bright and dark self-similar waves, as well as the trains of such waves propagating inside planar, graded-index waveguide amplifiers with self-focusing and self-defocusing Kerr nonlinearities, respectively. The new self-similar waves are supported by such media if the magnitudes of the gain and the linear inhomogeneity of the waveguide are adjusted to maintain a static quadratic phase chirp of each wave. The main difference between our self-similar solitary waves and those obtained in Refs. [11–13] is that the former can be realized in practice under much less restrictive conditions. The exact solutions discovered in Refs. [11–13] exist *only* if the parameters of the system, such as dispersion, nonlinearity, and gain in the optical case or nonlinearity and trapping potential in the matter-wave case, satisfy certain fairly restrictive conditions. In contrast, there is no restriction on the strength of diffraction and medium nonlinearity for the newly discovered self-similar solitary waves to exist [14]. This distinction is important because, while it may be possible, though difficult in practice, to simultaneously manage gain, dispersion, and nonlinearity of an optical fiber, it is much more difficult to attain any control over diffraction in a planar waveguide.

On the fundamental side, the significance of our results is that they provide a broad class of analytical solutions, describing solitonlike self-similar waves in amplifying, *inhomogeneous* nonlinear media. The intensity profiles of the new self-similar waves coincide with those of spatial solitons supported by homogeneous, passive media with the same type of nonlinearity. Thus, we expose a surprising connection between spatial self-similar waves in graded-index gain media and spatial solitons in homogeneous media with the same type of nonlinearity. We stress that inhomogeneity of the medium is a key condition for the existence of the present self-similar waves. Hence, the gained insights may also be relevant for other types of nonlinear waves in inhomogeneous and amplifying systems such as, for instance, matter waves in trapped Bose-Einstein condensates and atom lasers [15]. On a more practical side, we propose an efficient way the novel self-similar waves can be used to realize simultaneous amplification and focusing of bright and dark spatial solitons in the future all-optical networks.

We start by considering the propagation of a continuous-wave optical beam inside a planar, graded-index nonlinear waveguide amplifier with the refractive index

$$n = n_0 + n_1 x^2 + n_2 I. \tag{1}$$

Here the first two terms describe the linear part of the refractive index and the last term represents a Kerr-type nonlinearity of the waveguide amplifier. We assume $n_1 > 0$ so that, in the low intensity limit, the graded-index waveguide acts as a linear *defocusing* lens. This feature distinguishes our system from previously studied focusing waveguides and fibers with $n_1 < 0$ [16].

The nonlinear wave equation governing beam propagation in such a waveguide can be written as

$$i\frac{\partial u}{\partial z} + \frac{1}{2k_0}\frac{\partial^2 u}{\partial x^2} - \frac{ig}{2}u + \frac{1}{2}k_0 n_1 x^2 u + k_0 n_2 |u|^2 u = 0, \tag{2}$$

where g is the gain coefficient and $k_0 = 2\pi n_0/\lambda$, λ being the wavelength of the optical source generating the beam. If we introduce the normalized variables $X = x/w_0$, $Z = z/L_D$, $G = gL_D$, $U = (k_0|n_2|L_D)^{1/2}u$, where $L_D = k_0 w_0^2$ is the diffraction length associated with the characteristic transverse scale $w_0 = (2k_0^2 n_1)^{-1/4}$, we can rewrite Eq. (2) in a dimensionless form as

$$i\frac{\partial U}{\partial Z} + \frac{1}{2}\frac{\partial^2 U}{\partial X^2} - \frac{iG}{2}U + \frac{1}{2}X^2 U + \sigma|U|^2 U = 0. \tag{3}$$

Here $\sigma = \pm 1$, with the upper (lower) sign corresponding to a self-focusing (self-defocusing) nonlinearity of the waveguide.

The symmetry group analysis of Eq. (3) indicates that a self-similar wave solution to this equation ought to be sought in the form

$$U(X, Z) = \frac{1}{W(Z)}\Psi\left[\frac{X - X_c(Z)}{W(Z)}\right]\exp\{i[\Phi(X, Z)]\}, \tag{4}$$

where X_c is the position of the self-similar wave center.

To avoid a breakup of the beam due to an excessive nonlinear phase shift, we conjecture that the self-similar wave field must have an aberrationless spherical wave front $\Phi(X, Z)$ of the form

$$\Phi(X, Z) = \frac{1}{2}CX^2 + B(Z)X + \Theta(Z), \tag{5}$$

where the coefficient C is related to the wave front curvature; it is also a measure of the linear phase chirp imposed on the self-similar wave.

On substituting from Eqs. (4) and (5) into Eq. (3), collecting similar terms, and requiring that the real and imaginary parts of each term be separately equal to zero, we obtain, after some lengthy but straightforward algebra, a set of first-order differential equations for the width W, the coefficient B, and the beam center X_c. This set of equations is self-consistent only if the chirp parameter C and gain coefficient G obey the constraint

$$G = -C = 1. \tag{6}$$

The set of first-order differential equations can be readily solved to obtain the following expressions for W, B, and X_c:

$$W(Z) = W_0 e^{-Z}, \qquad B(Z) = B_0 e^{Z}, \tag{7}$$

$$X_c(Z) = X_0 e^{-Z} + B_0 \sinh(Z), \tag{8}$$

where W_0, B_0, and X_0 are initial values of the corresponding parameters. Furthermore, the self-similar wave profile $\Psi(\zeta)$ and the phase factor $\Theta(Z)$ are found to satisfy

$$\Psi''_{\zeta\zeta} = 2\beta\Psi - 2\sigma\Psi^3, \tag{9}$$

$$\Theta'_Z = \beta/W^2 - B^2/2, \tag{10}$$

where $\zeta = [X - X_c(Z)]/W(Z)$ is a similarity variable and β is a propagation constant to be determined by solving the eigenvalue problem posed by Eq. (9).

It is instructive to observe that even though the dynamics of the self-similar wave parameters, governed by Eqs. (7) and (8), is the same for propagation in linear and nonlinear media, the presence of the nonlinearity is essential for physical self-similar waves to exist. Indeed, it follows from Eq. (9) that, in the absence of the last term describing the nonlinearity, no bound solutions exist to this equation.

The condition $G = gL_D = 1$ in Eq. (6) is equivalent to requiring

$$g = \sqrt{2n_1}, \tag{11}$$

and it provides a necessary condition for the self-similar wave existence. Indeed, it follows from Eq. (11) that the characteristic longitudinal spatial scale associated with the amplification, $L_A = 1/g$, is completely determined by the value of the linear defocusing parameter n_1. Consequently, there is no *independent* characteristic longitudinal scale describing wave motions of our system, which is an unambiguous signature of the existence of a self-similar regime [1]. At this point, it is convenient to separately consider the cases of self-focusing and self-defocusing nonlinearities.

Self-focusing nonlinearity, $\sigma = 1$.—The analysis of Eq. (9) with $\sigma = 1$ reveals that there exists a bright soliton-like sech-profile solution for $\beta = 1/2$. On integrating Eqs. (9) and (10) with this value of β, we obtain a family of *bright self-similar waves*, with the field of each member given by

$$U_B(X, Z) = \frac{1}{W(Z)}\operatorname{sech}\left[\frac{X - X_c(Z)}{W(Z)}\right]\exp(-iX^2/2) \times \exp\{i[B(Z) + \Theta_B(Z)]\}. \tag{12}$$

Here the width and the position of the center of any bright self-similar wave are specified by Eqs. (7) and (8) and the accumulated phase $\Theta_B(z)$ is given by

$$\Theta_B(Z) = \frac{(1 - W_0^2 B_0^2)}{4W_0^2}e^{2Z}. \tag{13}$$

Further analysis of Eq. (9) indicates that for any positive $\beta \neq 1/2$, the new self-similar waves have the form of *periodic trains* such that

$$U_T(X,Z) = \frac{S_+}{W(Z)}\mathrm{dn}\left\{S_+\left[\frac{X - X_c(Z)}{W(Z)}\right]; k\right\} \times \exp\{i[-X^2/2 + B(Z) + \Theta_B(Z)]\}, \quad (14)$$

where $\mathrm{dn}(x;k)$ is a Jacobi elliptic function with the elliptic modulus $k = [(S_+^2 - S_-^2)/S_+^2]^{1/2}$. The parameters $S_\pm$ are defined as $S_\pm = \beta \pm (\beta^2 - |\gamma|)^{1/2}$, where γ is a real negative constant such that $0 < -\gamma < \beta$. Equation (14) describes a periodic train of self-similar waves, whose amplitude grows exponentially and whose period decays exponentially with Z as $4K(k)W_0e^{-Z}/S_+$, where $K(k)$ is a complete elliptic integral of the first kind. Of course, physical, finite-power realizations of such periodic self-similar wave solutions do not extend to infinity along the X direction; rather they are matched to the exponentially decaying radiation modes at the edges of the waveguide.

Self-defocusing nonlinearity, $\sigma = -1$.—A similar analysis of Eqs. (9) and (10) in the case of a defocusing nonlinearity reveals that a *dark self-similar wave* family also exists for $\beta = 1$. The optical field associated with each member of this family can be written as

$$U_D(X,Z) = \frac{1}{W(Z)}\tanh\left[\frac{X - X_c(Z)}{W(Z)}\right]\exp(-iX^2/2) \times \exp\{i[B(Z) + \Theta_D(Z)]\}. \quad (15)$$

Again, the width and the position of the center of any dark self-similar wave are given by Eqs. (7) and (8), and the phase $\Theta_D(z)$ is given by

$$\Theta_D(Z) = \frac{(2 - W_0^2B_0^2)}{4W_0^2}e^{2Z}. \quad (16)$$

The newly discovered families of bright and dark self-similar waves are completely determined by three free parameters—the values of the width W_0, the beam center X_0, and the phase-shift coefficient B_0 in the source plane. We stress that these self-similar waves represent *exact, nonperturbative solutions* to nonlinear wave Eq. (3). Thus, they are drastically different from the well-known results of soliton perturbation theory, which predicts the existence of solitons with slowly (adiabatically) varying width, velocity, and the phase under the influence of small perturbations [17].

Another interesting feature of the novel self-similar waves is associated with the nontrivial dynamics of the solitary wave center: The analysis of Eq. (8) reveals that if $\mathrm{sign}(X_0) = \mathrm{sign}(B_0)$ and $|X_0| > |B_0|$, the magnitude of the wave center coordinate attains its minimum value at a certain distance; otherwise it increases monotonically. This behavior of the beam center is clearly seen in Figs. 1 and 2, where we display analytical solutions for the bright and dark self-similar waves under the initial conditions such that in Figs. 1(a) and 2(a) $|X_0| > |B_0|$ and in Figs. 1(b) and 2(b) $|X_0| \leq |B_0|$, respectively. To

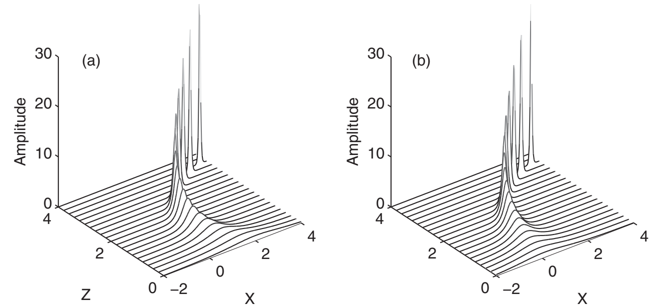

FIG. 1. Evolution of a bright self-similar wave for (a) $X_0 = 2$ and (b) $X_0 = 0.1$. In both cases, $B_0 = 0.2$ and $W_0 = 1$.

explain such a behavior, we use Eqs. (6) and (7) to rewrite the expression for the phase $\Phi(X,Z)$ as

$$\Phi(X,Z) = -\tfrac{1}{2}(X - B_0e^Z)^2 + \tilde{\Theta}(Z). \quad (17)$$

It follows from Eqs. (8) and (17) that, even if the beam-center position X_0 does not coincide with the phase-curvature center position B_0 initially at $Z = 0$, both centers evolve in unison at sufficiently large Z. A qualitative analysis of the coupled dynamics of the phase and amplitude of the self-similar wave indicates that the beam-center and phase-curvature center must propagate in concert for the wave to maintain constant intensity profile. It follows that if $|X_0| > |B_0|$, the magnitude of X_c has to decrease during the initial transition period to ensure that it evolves in unison with the phase-curvature center. This feature explains the different evolution scenarios of the beam center exhibited in the two parts of Figs. 1 and 2 for the bright and dark self-similar waves, respectively.

To address the stability issue, we solved Eq. (3) numerically with the split-step Fourier method [18]. Figure 3 shows the amplitude of self-similar waves for $Z = 1$, 2, and 2.5 for $\sigma = 1$, $G = 1$, $X_0 = 5$, and $B_0 = 0.2$. The analytical solution in Eq. (12) is plotted for comparison as a dotted line. The two solutions nearly coincide at $Z = 1$ but begin to differ in the beam wings for $Z > 1.5$ because of the continuum radiation emitted by such waves. The radiation level is near 10^{-3} at $Z = 2$ but increases rapidly to exceed 1% at $Z = 3$ (corresponding to an intensity level

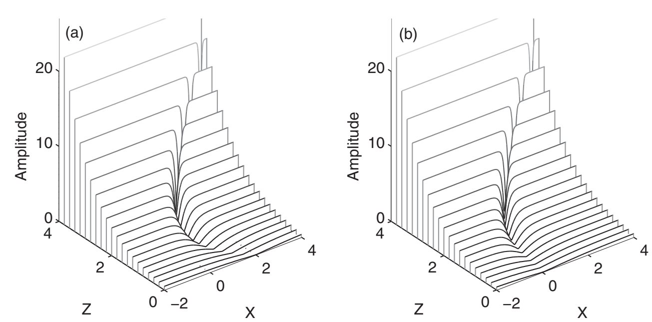

FIG. 2. Evolution of a dark self-similar wave for (a) $X_0 = 2$ and (b) $X_0 = 0.1$. In both cases, $B_0 = 0.2$ and $W_0 = 1$.

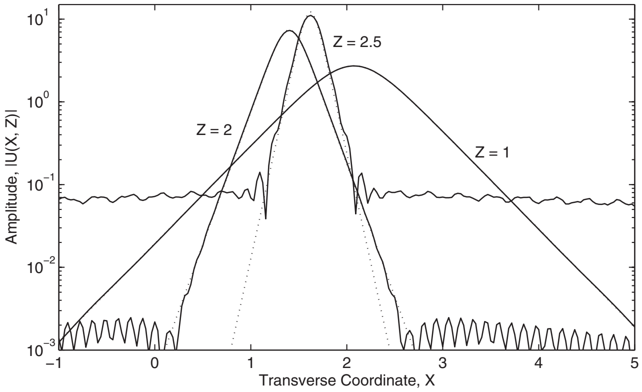

FIG. 3. Numerical (solid lines) and analytical (dotted lines) profiles of self-similar bright solitons for Z in the range of 1 to 2.5 when $X_0 = 5$, $W_0 = 1$, and $B_0 = 0.2$. The oscillating tail represents continuum radiation emitted by the soliton.

of 0.01%). Since the bright and dark self-similar waves found in this Letter are stable over a few diffraction lengths, they may still be useful for practical applications.

In particular, the solitonlike nature of our self-similar waves hints to the possibility of designing a waveguide soliton amplifier. The proposed device is expected to operate as follows. A linear phase chirp is imprinted on a fundamental bright (or dark) spatial soliton using an appropriate phase mask placed at the entrance to the graded-index waveguide. If the amplifier gain g satisfies the condition given in Eq. (11), the entering phase-chirped spatial soliton propagates inside the amplifier as a self-similar wave found in this Letter and is thus compressed as it is amplified, while preserving its shape. At the exit of the amplifier, a second phase mask is used to remove the phase chirp. The resulting beam is an amplified and focused bright (or dark) soliton that may be useful for future all-optical networks.

Before concluding, we consider a suitable example and focus on a 5 cm long planar silicon waveguide, pumped optically to amplify an input beam by a factor of 10 through the Raman gain ($g \simeq 0.46\ \mathrm{cm}^{-1}$). The waveguide is only 1 μm thick and confines input beam in the y direction. No such confinement occurs in the x direction along which n varies as indicated in Eq. (11). To satisfy Eq. (1), the refractive index must be graded with $n_1 \simeq 0.1\ \mathrm{cm}^{-2}$, which lead to $w_0 \approx 70\ \mu$m near 1.55 μm. The diffraction length for this value of w_0 is about 2 cm. If we use $n_2 = 6 \times 10^{-14}\ \mathrm{cm}^2/W$, the required peak intensity is $100\ \mathrm{MW/cm}^2$, translating into input power levels ~100 W. Such power levels are easily realized under quasicontinuous conditions (relatively broad pulses) for which our theory remains applicable.

In conclusion, we have discovered and analytically described a wide class of spatial solitonlike self-similar waves which can propagate in graded-index, nonlinear waveguide amplifiers. Our results shed light on the interesting connection between self-similar waves and solitons existing in inhomogeneous and homogeneous nonlinear media, respectively. Using novel self-similar waves, we propose a potentially powerful method for the amplification and focusing of spatial solitons to overcome inevitable energy losses from which such solitons will suffer while performing multiple functions in futuristic all-optical networks.

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