Phase-space quality factor for ultrashort pulsed beams

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We derive an expression for the lower bound of the phase-space beam quality factor $\mathcal{M}^2$ of an ultrashort pulse. We show that the condition $\mathcal{M}^2 \geq 1$ does not, in general, hold for such pulsed beams. Rather, the minimum value of $\mathcal{M}^2$ depends on the pulse spectrum. We also show that $\mathcal{M}^2$ attains minimum only for pulsed beams whose spot size varies with frequency as $\omega^{-1/2}$, and that the radial intensity profile of the best-quality pulsed beam can be non-Gaussian. © 2008 Optical Society of America

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In many applications, it is desirable to have highly directional beams with the smallest possible spot size at the source. However, because of an inverse (Fourier-transform) relationship between the spot size of the beam and its far-field angular spread [1], it is only possible to trade one off for the other. To characterize the beam quality, a dimensionless quantity, known as the $\mathcal{M}^2$ factor, is often used in practice [2]. Mathematically, this factor is related to the phase-space product of the root-mean-square (rms) beam width and the rms value of the far-field angle [3–7] and is defined such that $\mathcal{M}^2=1$ for an ideal Gaussian beam whose beam waist is located at the source plane.

Even though $\mathcal{M}^2$ factor was originally introduced in the context of continuous-wave (cw) beams, it is often used to characterize the quality of pulsed beams as well. In the case of relatively long pulses such as those emitted by Q-switched lasers, a time-dependent $\mathcal{M}^2$ factor is sometimes employed [4–6]. Yet, such an approach is both fundamentally unsound and impractical in the case of femtosecond pulses emitted by modern mode-locked lasers [8]. For ultrashort pulses lasting for only a few optical cycles, the spectral bandwidth becomes comparable in magnitude to the carrier frequency of the pulse [9]. Although the properties of such pulsed beams, propagating in free space or linear dispersive media, have been extensively investigated in recent years [10–16], a rather subtle question of how to define the corresponding $\mathcal{M}^2$ factor has so far not been addressed. A related basic question is: What is the radial intensity profile associated with the best-quality ultrashort pulsed beam?

In this Letter, we propose a definition for the phase-space quality factor of ultrashort pulsed beams. We show that the magnitude of such a pulse-beam quality factor $\mathcal{M}^2$ has a minimum value that depends on the characteristics of the pulse source spectrum. We also demonstrate that the radial intensity profile of the best-quality pulsed beam can significantly deviate from a Gaussian.

In the following, it is convenient to work in the space-frequency representation by introducing a spectral decomposition of the optical field $V(\boldsymbol{\rho},t)$ of the pulse viz., $U(\boldsymbol{\rho},\omega)=\int_{-\infty}^{\infty}\mathrm{d}tV(\boldsymbol{\rho},t)\exp(i\omega t)$. The rms width of the pulsed beam at the source plane can then be defined as

$$\langle\rho^2\rangle=\frac{\int_0^\infty \mathrm{d}\omega\int d^2\rho\rho^2|U(\boldsymbol{\rho},\omega)|^2}{\int_0^\infty \mathrm{d}\omega\int d^2\rho|U(\boldsymbol{\rho},\omega)|^2},\qquad(1)$$

where $|U(\boldsymbol{\rho},\omega)|^2$ is the density of the energy spectrum $S(\omega)$, which was predicted to be a directly measurable quantity for femtosecond laser pulses [17]. For fully coherent pulsed beams, $S(\omega)$ is defined by the relation

$$S(\omega)\equiv\int \mathrm{d}^2\rho|U(\boldsymbol{\rho},\omega)|^2.\qquad(2)$$

The rms value of the far-field angle can be defined in a similar manner using the concept of the radiant intensity. More precisely,

$$\langle s_\perp^2\rangle=\frac{\int_0^\infty \mathrm{d}\omega\int d^2s_\perp s_\perp^2,J(k\mathbf{s}_\perp,\omega)}{\int_0^\infty \mathrm{d}\omega\int d^2s_\perp J(k\mathbf{s}_\perp,\omega)},\qquad(3)$$

where the radiant intensity, $J(k\mathbf{s}_\perp,\omega)$, is given by [1]

$$J(k\mathbf{s}_\perp,\omega)=(2\pi k)^2\cos^2\theta|\tilde{U}(k\mathbf{s}_\perp,\omega)|^2.\qquad(4)$$

Here, $k=\omega/c$ is the propagation constant, $\mathbf{s}_\perp$ is a two-dimensional (2D) vector projection onto the source plane of a [three-dimensional (3D)] unit vector $\mathbf{s}$ pointing from the source to the far zone, and $\tilde{U}(k\mathbf{s}_\perp,\omega)$ is the 2D spatial Fourier transform of $U(\boldsymbol{\rho},\omega)$.

In most practical situations, one deals with paraxial sources whose angular distribution peaks sharply along the z direction such that $\cos\theta\simeq 1$ and $|\mathbf{s}_\perp|=\sin\theta\simeq\theta$. Hence, we can extend the limits of in-

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tegration over $k\mathbf{s}_\perp$ to cover the entire 2D Fourier plane. It then follows from Eqs. (3) and (4) that the angular spread of a pulse beam generated by a paraxial source is given by

$$\langle s_\perp^2 \rangle = \frac{\int_0^\infty \mathrm{d}\omega \int d^2(ks_\perp) s_\perp^2 |\tilde{U}(k\mathbf{s}_\perp,\omega)|^2}{\int_0^\infty \mathrm{d}\omega \int d^2(ks_\perp) |\tilde{U}(k\mathbf{s}_\perp,\omega)|^2}. \quad (5)$$

To obtain a lower bound for the phase-space product of pulsed beams, we generalize the approach of [18] by considering the following functional:

$$\Phi(\alpha) = \frac{1}{W}\int_0^\infty \mathrm{d}\omega \int d^2\rho (\mathbf{f}^* \cdot \mathbf{f}) \geq 0, \quad (6)$$

where the vector $\mathbf{f}$ is defined as

$$\mathbf{f} \equiv \boldsymbol{\rho} U(\boldsymbol{\rho},\omega) + \alpha(\omega)\nabla U(\boldsymbol{\rho},\omega), \quad (7)$$

α is any *real* function of ω, and the total energy W of the pulse is expressed as

$$W = \int_0^\infty \mathrm{d}\omega \int d^2\rho |U|^2 = \int_0^\infty d\omega \int \mathrm{d}^2(ks_\perp)|\tilde{U}|^2. \quad (8)$$

On observing that

$$\int d^2\rho \nabla U^* \nabla U = \int d^2(ks_\perp) k^2 s_\perp^2 |\tilde{U}|^2, \quad (9)$$

which follows from the properties of Fourier transforms, we can cast the inequality in Eq. (6) into the form

$$\langle \rho^2 \rangle - 2\alpha_0 F_s(\omega_0) + \alpha_0^2 k_0^2 \langle s_\perp^2 \rangle \geq 0, \quad (10)$$

provided that α obeys the scaling relation $\alpha(\omega) = \alpha_0(\omega_0/\omega)$. In Eq. (10), F_s represents a spectral form factor, defined as

$$F_s(\omega_0) \equiv \frac{\int_0^\infty \mathrm{d}\omega \left(\frac{\omega_0}{\omega}\right) S(\omega)}{\int_0^\infty \mathrm{d}\omega S(\omega)}, \quad (11)$$

where ω_0 is the frequency at which the source spectrum attains maximum.

We now introduce the $\mathcal{M}^2$ factor in the form of a phase-space product as

$$\mathcal{M}^2 \equiv k_0 \sqrt{\langle s_\perp^2 \rangle \langle \rho^2 \rangle}, \quad (12)$$

and we propose to use it for assessing the phase-space quality of any pulsed beam. The advantage of our definition over the previously introduced time-resolved one [6] is that the former represents a time-independent scalar quantity characterizing the whole pulse. It follows at once from Eq. (10) that there exists a lower bound on the magnitude of $\mathcal{M}^2$ such that

$$\mathcal{M}^2 \geq F_s(\omega_0). \quad (13)$$

Inequality (13) is a key result of this Letter. It shows that the quality factor of a pulsed beam, in general, depends on the spectral characteristics of the corresponding source. As required, the new inequality reduces to the standard result $\mathcal{M}^2 \geq 1$ for cw or quasi-cw sources whose bandwidth is narrow enough, $S(\omega) \propto \delta(\omega - \omega_0)$.

For ultrashort pulses with a relatively wide spectrum, the minimum value of $\mathcal{M}^2$ exceeds 1. Consider, for example, a Gaussian pulse whose spectrum is of the form $S(\omega) \propto \exp[-(\omega-\omega_0)^2/\delta^2]$, where δ is a measure of the spectral bandwidth. It is seen in Fig. 1 that F_s monotonically increases as a function of the ratio δ/ω_0, exceeding 2 for $\delta/\omega_0 > 0.7$. Clearly, $\mathcal{M}^2$ close to 1 should not be expected for pulsed beams whose spectral bandwidth becomes comparable to their carrier frequencies. It should be stressed that the choice of a Gaussian spectrum becomes inappropriate as $\delta/\omega_0 \to 1$ because $S(\omega)$ does not vanish at $\omega=0$, as required of any physically realizable spectrum.

To model more accurately the ultrashort pulse spectrum, we consider a realistic femtosecond pulse source [9]. The energy spectrum of any such source should exhibit (a) an infrared cutoff, (b) a degree of asymmetry with a high-frequency tail, and (c) a well-defined peak at a certain frequency. We propose the following phenomenological model for such a spectrum:

$$S(\omega) \propto \begin{cases} (\omega - \omega_*)^q e^{-a\omega} & \text{if } \omega \geq \omega_* \\ 0 & \text{if } \omega \leq \omega_* \end{cases}, \quad (14)$$

where ω_* is the infrared cutoff frequency and the parameter $a = q/(\omega_0 - \omega_*)$ is chosen such that the spectrum peaks at ω_0. The real positive exponent q and the ratio ω_0/ω_* describe the asymmetry of the spectrum. In Fig. 2 we display pulse spectra (top) for $q = 1$, and the corresponding form factors (bottom) as functions of q for four values of the ratio ω_0/ω_*. When this ratio is relatively large, F_s exceeds 2. Values closer to 1 are realized for $\omega_0/\omega_* < 10$, and F_s becomes smaller than 1 when ω_0/ω_* is below 6.

An important question is under what conditions $\mathcal{M}^2$ of a pulsed beam attains its minimum value F_s.

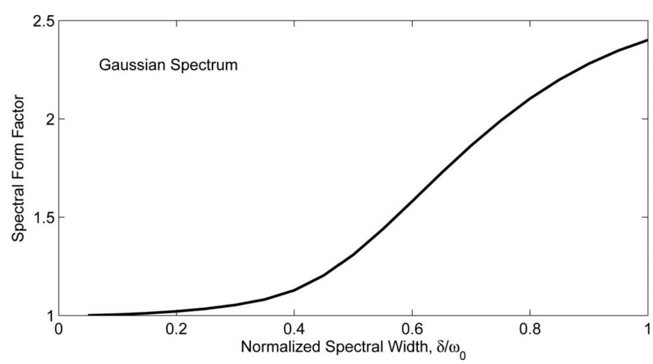

Fig. 1. Spectral form factor as a function of δ/ω_0 for pulses whose spectrum is centered at frequency ω_0.

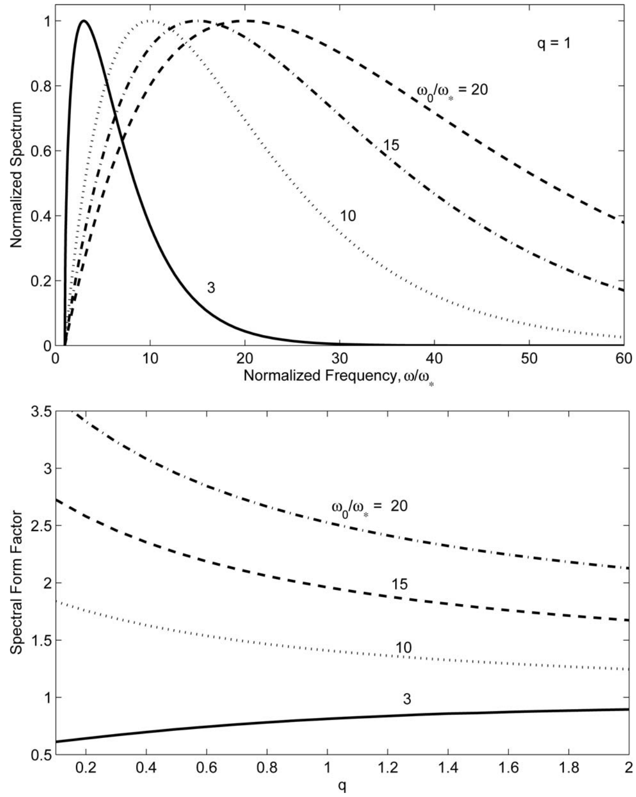

Fig. 2. Pulse spectra as functions of ω/ω_* for $q=1$, and form factors as functions of q for four values of ω_0/ω_*.

The field distribution of such a beam can be determined from Eqs. (6) and (7) by noting that **f** should be set to zero, or

$$\boldsymbol{\rho} U(\boldsymbol{\rho},\omega) + \alpha(\omega)\nabla U(\boldsymbol{\rho},\omega) = 0. \tag{15}$$

A straightforward integration of this equation yields

$$U(\rho,\omega) = A(\omega)\exp\left[-\frac{\rho^2}{2\sigma^2(\omega)}\right], \tag{16}$$

where $A(\omega)$ is the spectral density of the pulsed beam on the axis, and we have introduced the (frequency-dependent) beam width $\sigma(\omega)$ by the expression

$$\sigma(\omega) = \sigma_0(\omega_0/\omega)^{1/2}. \tag{17}$$

As is seen from Eq. (16), the field $U(\rho,\omega)$ does not factorize into spectral and spatial parts for the best phase-space quality pulsed beam. To study the spatial distribution of the energy in the pulsed beam, we determine its radial intensity profile, defined as

$$I(\boldsymbol{\rho}) \equiv \int_0^\infty \mathrm{d}\omega |U(\boldsymbol{\rho},\omega)|^2. \tag{18}$$

Substituting from Eq. (16) into Eq. (18) and eliminating A in favor of the measurable energy spectrum, we obtain

$$I(\rho) = \frac{1}{4\pi}\int_0^\infty \mathrm{d}\omega \frac{S(\omega)}{\sigma^2(\omega)} \exp\left[-\frac{\rho^2}{2\sigma^2(\omega)}\right]. \tag{19}$$

In Fig. 3, we show the radial intensity profile $I(\rho)$ for

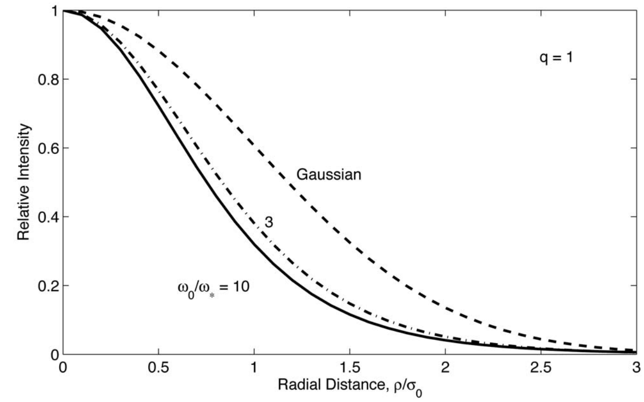

Fig. 3. Radial intensity profile for two values of ω_0/ω_*. The dashed curve shows a Gaussian profile of width σ_0 for comparison.

two values of ω_0/ω_* for ultrashort pulses with the spectrum given in Eq. (14). A Gaussian profile for a quasi-monochromatic beam centered at ω_0 is also presented for comparison. Clearly, the beam profile deviates considerably from a Gaussian shape for such broadband pulse spectra.

In conclusion, we have shown that the condition $\mathcal{M}^2 \geq 1$ does not hold for optical beams consisting of ultrashort pulse trains. Rather, the minimum value of $\mathcal{M}^2$ depends on the pulse spectrum. It exceeds 1 in most cases of practical interest, but $\mathcal{M}^2 < 1$ is possible for certain pulse spectra. We also show that the radial beam profile of the best phase-space quality pulse can significantly deviate from a Gaussian shape.

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