On a class of electromagnetic diffraction-free beams

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We present a general theory of electromagnetic diffraction-free beams composed of uncorrelated Bessel modes. Our approach is based on the direct application of the nonnegativity constraint to the cross-spectral density tensor describing the electromagnetic field distribution. The field correlation properties are most conveniently derived in the spatial frequency domain, where the angular spectrum takes on the form of an infinitely thin ring. We also present several examples, including a vector generalization of the recently introduced dark and antidark diffraction-free beams.

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1. INTRODUCTION

The existence of diffraction-free beams was pointed out by Sheppard and Wilson in 1978 [1]. It seems, however, that the physics community became aware of the associated possibilities only after the paper by Durnin *et al.* in 1987 [2]. This triggered a flurry of activity resulting in thorough studies of various classes of spatial, temporal and spatiotemporal nondiffracting fields [3–12]. While the majority of diffraction-free beams examined to date have been fully spatially coherent [2–8], there has been growing interest in partially spatially coherent diffraction-free beams [9–11]. In particular, linear cousins of dark and antidark optical solitons [13], the so-called dark and antidark diffraction-free beams, have been recently introduced, and their partially coherent nature has been explained in[11].

To our knowledge, however, research on partially coherent diffraction-free beams has so far been essentially restricted to the scalar treatment (for some notable exceptions, see[14–16] and references therein).

The purpose of the present paper is to develop a general framework within which a multitude of electromagnetic partially correlated diffraction-free beams can be treated. As a particular illustration of our approach, we provide a vectorial generalization of the scalar dark and antidark diffraction-free (DADDF) beams. We also determine a full electromagnetic mode expansion of the novel beams and demonstrate that the vector modes are given by uncorrelated superpositions of suitably polarized Bessel beams.

This work is organized as follows. In Section 2, we outline the necessary elements of coherence theory and stress the relevant terminology. This is followed by a brief description of recently introduced DADDF beams, which naturally evolves into their electromagnetic generalization in Section 3. In Section 4, we present and discuss several examples of novel electromagnetic diffraction-free beams composed of uncorrelated Bessel modes. We summarize our findings in Section 5.

2. PRELIMINARIES

In this section we recall a few relevant concepts and tools of coherence theory. For a more detailed discussion the reader is referred to such textbooks as [17,18]. Hereafter we restrict our attention to planar sources, which can be either primary or secondary ones. In many practical situations, a scalar description of optical coherence properties of the fields generated by such sources may well be sufficient. Scalar coherence theory can be formulated in either the space–time or the space–frequency domain [17,18]. In the latter case, the scalar description of a stochastic source is based on the cross-spectral density (CSD) defined as

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \langle V^*(\boldsymbol{\rho}_1,\omega)V(\boldsymbol{\rho}_2,\omega)\rangle, \tag{1}$$

where $V(\boldsymbol{\rho},\omega)$ is an ensemble representative of a scalar optical field of frequency ω at the position specified by the vector $\boldsymbol{\rho}$ in the source plane. In the following, we shall omit any explicit dependence on ω. The angular brackets denote the average over an ensemble of statistical realizations of the field, and the asterisk stands for a complex conjugate. It can be shown that W must be nonnegative definite, implying that for any—finite or denumerable—set of arbitrarily chosen position vectors $\boldsymbol{\rho}_n$ and complex constants a_n, the following quadratic form

$$Q = \sum_{n,m} W(\boldsymbol{\rho}_n,\boldsymbol{\rho}_m)a_n^* a_m, \tag{2}$$

has to be nonnegative. The continuous version of this quadratic form reads

$$Q = \int \int \mathrm{d}^2\boldsymbol{\rho}_1 \mathrm{d}^2\boldsymbol{\rho}_2 W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) f^*(\boldsymbol{\rho}_1) f(\boldsymbol{\rho}_2), \tag{3}$$

which has to be nonnegative for any choice of the function $f(\boldsymbol{\rho})$. Obviously, Eq. (2) can be considered a particular case of Eq. (3) if the function $f(\boldsymbol{\rho})$ reduces to a set of Dirac delta functions.

The concept of coherent modes [19] is at the heart of scalar coherence theory. To introduce the modes, we consider a homogeneous Fredholm integral equation of the second kind,

$$\int \mathrm{d}^2\boldsymbol{\rho}_1 W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) u(\boldsymbol{\rho}_1) = \lambda u(\boldsymbol{\rho}_2), \tag{4}$$

where the integral is extended across the source plane. Under the assumption that $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ is a Hilbert–Schmidt kernel, the eigenfunctions, say $u_n(\boldsymbol{\rho})$, form a set of orthonormal functions. Because of the nonnegative definiteness of the kernel W, the associated eigenvalues, say λ_n, are nonnegative. Furthermore, CSD can be represented by the following Mercer-type series:

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \sum_n \lambda_n u_n^*(\boldsymbol{\rho}_1) u_n(\boldsymbol{\rho}_2). \tag{5}$$

Equation (5) is called the modal expansion of the CSD, and the functions $u_n(\boldsymbol{\rho})$ are referred to as the source modes. It is important to note that each mode represents a spatially coherent field. The physical meaning of Eq. (5) is that the various modes are superposed in an uncorrelated way, each mode carrying power proportional to the associated eigenvalue. This gives rise to a partially coherent field distribution except when only one of the eigenvalues differs from zero.

Since the mode and eigenvalue determination involves solving the Fredholm integral equation, considerable difficulties arise in finding explicit expressions for these quantities. As a matter of fact, closed-form modal expansions are available, but for a limited number of cases (see, for instance, [20–24]). On the other hand, detailed knowledge of the spatial mode structure may be irrelevant for solving certain problems; the mere existence of the modes suffices.

Let us now proceed to the electromagnetic case. As long as the field generated by the source is beamlike, coherence and polarization properties can be described either in the space–time domain by the so-called beam coherence-polarization (BCP) matrix [25,26] or in the space–frequency domain by the CSD matrix [18,27,28]. Here, we will employ the latter, which is defined as

$$\hat{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \begin{bmatrix} W_{xx}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) & W_{xy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) \\ W_{yx}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) & W_{yy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) \end{bmatrix}, \tag{6}$$

where the matrix elements are evaluated as

$$W_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \langle E_\alpha^*(\boldsymbol{\rho}_1) E_\beta(\boldsymbol{\rho}_2)\rangle. \tag{7}$$

Hereafter, we will use the subscripts α and β to label the Cartesian components x or y of the field. It is worthwhile to note that when $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2$, the CSD matrix has the same form as the ordinary polarization matrix [29]. The matrix, however, can change form one point to another, so that one speaks of a local polarization matrix, say $\hat{P}(\boldsymbol{\rho})$, determined by

$$\hat{P}(\boldsymbol{\rho}) \equiv \hat{W}(\boldsymbol{\rho},\boldsymbol{\rho}) = \begin{bmatrix} W_{xx}(\boldsymbol{\rho},\boldsymbol{\rho}) & W_{xy}(\boldsymbol{\rho},\boldsymbol{\rho}) \\ W_{yx}(\boldsymbol{\rho},\boldsymbol{\rho}) & W_{yy}(\boldsymbol{\rho},\boldsymbol{\rho}) \end{bmatrix}. \tag{8}$$

The quadratic form to be considered for the present case is

$$Q = \sum_{\alpha,\beta} \sum_{n,m} W_{\alpha\beta}(\boldsymbol{\rho}_{1n},\boldsymbol{\rho}_{2m}) a_n^* b_m. \tag{9}$$

Here, the two sets of arbitrary constants $\{a_n\}$ and $\{b_m\}$ and those of the position vectors $\{\boldsymbol{\rho}_{1n}\}$ and $\{\boldsymbol{\rho}_{2m}\}$ are involved. The continuous version of Q reads

$$Q = \sum_{\alpha,\beta} \int \int \mathrm{d}^2\boldsymbol{\rho}_1 \mathrm{d}^2\boldsymbol{\rho}_2 W_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) f_\alpha^*(\boldsymbol{\rho}_1) f_\beta(\boldsymbol{\rho}_2), \tag{10}$$

where $f_\alpha(\boldsymbol{\rho})$ is an arbitrary function. Nonnegativeness of the CSD matrix requires that Q never become negative.

In order to find the modal expansion of the CSD matrix, we have to solve the equations

$$\sum_\beta \int \mathrm{d}^2\boldsymbol{\rho}_1 W_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) u_\beta(\boldsymbol{\rho}_1) = \lambda u_\alpha(\boldsymbol{\rho}_2). \tag{11}$$

Equations (11) constitute a system of two coupled homogeneous Fredholm integral equations admitting a finite or denumerable set of eigenvectors $\boldsymbol{u}_n(\boldsymbol{\rho})$ and eigenvalues λ_n. Each eigenvector has two Cartesian components, say $u_{nx}(\boldsymbol{\rho})$ and $u_{ny}(\boldsymbol{\rho})$. They can be arranged either in a row or in a column vector. We shall use the second choice so that the eigenvector can be thought of as a Jones vector [29].

The CSD matrix can then be expressed by the series

$$\hat{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \sum_n \lambda_n \boldsymbol{u}_n(\boldsymbol{\rho}_1) \boldsymbol{u}_n^\dagger(\boldsymbol{\rho}_2), \tag{12}$$

where the dagger denotes the Hermitian conjugate. Obviously, finding eigenvectors and eigenvalues is even more challenging than in the scalar case. For an example of closed-form solutions, see [30].

A comment about terminology is in order. In the following we shall be concerned with the existence of correlations among fluctuating quantities. In this context, each term in the sum on the r.h.s. of Eq. (12) represents a CSD matrix of a single mode. All Cartesian components of the electric field of any given mode are mutually fully correlated. On the other hand, any pair of Cartesian components pertaining to different modes are completely uncorrelated so that the CSD becomes a superposition of the number of uncorrelated contributions. The overall level of field correlations will then depend on the structure of the modes and the eigenvalue distribution.

In the next sections, we will extensively use the Fourier transform (FT). We will use it in its symmetrical form. Accordingly, for a typical function $s(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$, we shall evaluate its FT, to be denoted by a tilde, as

$$\tilde{s}(\boldsymbol{p}_1,\boldsymbol{p}_2) = \int\int \mathrm{d}^2\rho_1 \mathrm{d}^2\rho_2 s(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)\exp[-2\pi\mathrm{i}(\boldsymbol{p}_1\cdot\boldsymbol{\rho}_1+\boldsymbol{p}_2\cdot\boldsymbol{\rho}_2)], \tag{13}$$

with a similar (and simpler) expression holding for functions with a single argument.

It is useful to note that, using the FT, Eq. (10) can be converted to the spatial frequency domain as follows:

$$Q = \sum_{\alpha,\beta} \int\int \mathrm{d}^2p_1 \mathrm{d}^2p_2 \widetilde{W}_{\alpha\beta}(\boldsymbol{p}_1,-\boldsymbol{p}_2)\tilde{f}^*_\alpha(\boldsymbol{p}_1)\tilde{f}_\beta(\boldsymbol{p}_2). \tag{14}$$

The nonnegative definite character of $\hat{W}$ can then be assessed in terms of FTs. More explicitly, if we denote by $\hat{\mathcal{W}}(\boldsymbol{p}_1,\boldsymbol{p}_2)$ the matrix whose elements are $\widetilde{W}_{\alpha\beta}(\boldsymbol{p}_1,-\boldsymbol{p}_2)$, we can say that the nonnegative definiteness of $\hat{W}$ implies that of $\hat{\mathcal{W}}$, and vice versa.

3. DIFFRACTION-FREE BEAMS

We shall begin by recalling scalar diffraction-free beams. In the coherent case, the simplest of such beams is described by an optical field with the structure

$$\Phi_n(\boldsymbol{\rho}) = J_n(\kappa\rho)\exp(\mathrm{i}n\phi), \tag{15}$$

where J_n is the Bessel function of the first kind and order n and κ is a scaling parameter. The variables ρ,ϕ are the polar coordinates associated with the vector $\boldsymbol{\rho}$. Beams of this type are called Bessel beams. More generally, any, finite or denumerable, linear combination of the form

$$V(\boldsymbol{\rho}) = \sum_n a_n\Phi_n(\boldsymbol{\rho}) \tag{16}$$

will also produce a coherent diffraction-free beam, provided only that the sum in Eq. (16) converges for any $\boldsymbol{\rho}$. The optical field in Eq. (16) is said to be fully spatially coherent because it is obtained by a linear superposition of Bessel modes. It is worth noting that the FT of any such beam is concentrated on an infinitely thin annulus in the spatial frequency plane. This stems from the fact that the FT of any Bessel beam in polar coordinates (p,θ) reads

$$\tilde{\Phi}_n(p,\theta) = (-\mathrm{i})^n\delta[p-\kappa/(2\pi)]\exp(\mathrm{i}n\theta), \tag{17}$$

where δ denotes the Dirac delta function. Consequently, the FT of V is also concentrated along the circle $p=\kappa/(2\pi)$.

In its simplest form, the extension from a coherent to a partially coherent case is accomplished by combining the Bessel modes in an uncorrelated way. This gives rise to the CSD of the form of Eq. (5) with the eigenfunctions u_n replaced by the Φ_n and nonnegative eigenvalues λ_n. In general, an arbitrary choice of λ_n's does not at all guarantee that the corresponding CSD will be expressed in a closed form. On the contrary, this happens for a few cases only. As a simple example, we may consider the situation of all equal eigenvalues. In this case, the CSD is proportional to $J_0(|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)$ [21]. Recently, a class of scalar diffraction-free partially coherent beams, called dark and antidark, has been introduced [11]. The CSD of such beams is given by the formula

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = J_0(\kappa|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|) + \chi J_0(\kappa|\boldsymbol{\rho}_1+\boldsymbol{\rho}_2|), \tag{18}$$

where χ is a *real* parameter with $\chi\in[-1,1]$. The modal expansion of such beams [11] is of the form of Eq. (5), with eigenfunctions given by Eq. (15). We can rewrite it as

$$W_{\mathrm{df}}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \sum_{n=-\infty}^{\infty}\lambda_n\Phi^*_n(\boldsymbol{\rho}_1)\Phi_n(\boldsymbol{\rho}_2), \tag{19}$$

where the subscript "df" stresses the diffraction-free nature of the beams and the eigenvalues are given by

$$\lambda_n = 1 + (-1)^n\chi. \tag{20}$$

The limiting case $\chi=-1$ can be termed "black" diffraction-free beams in analogy with the soliton case [11]. We remark in passing that the DADDF beam class can be enriched by considering uncorrelated superpositions of the form (19) with different values of the scaling parameter κ and, for each κ, arbitrary distributions of nonnegative eigenvalues λ_n.

We now wish to extend the scalar treatment to the electromagnetic case, having in mind, in particular, DADDF beams. To this end we assume that both diagonal elements W_{xx} and W_{yy} are equal to the scalar CSD W_{df} given by Eq. (19). With such an assumption, the field emitted by the vectorial source defined by $\hat{W}$ turns out to be indistinguishable, as far as only scalar measurements are concerned, from that emitted by the scalar source described by W_{df} in Eq. (19). The critical point is how to choose the off-diagonal element W_{xy} in order for the correlation matrix $\hat{W}$ to satisfy the nonnegative definiteness condition. As we shall see, this poses severe restrictions on the choice of W_{xy}. Let us write the quadratic form Q introduced by Eq. (9) as

$$Q = \int\int \mathrm{d}^2\rho_1\mathrm{d}^2\rho_2\{W_{df}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)[f^*(\boldsymbol{\rho}_1)f(\boldsymbol{\rho}_2)+g^*(\boldsymbol{\rho}_1)g(\boldsymbol{\rho}_2)] + 2\,\Re[W_{xy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)f^*(\boldsymbol{\rho}_1)g(\boldsymbol{\rho}_2)]\}, \tag{21}$$

where we let $f_x(\boldsymbol{\rho})=f(\boldsymbol{\rho})$ and $f_y(\boldsymbol{\rho})=g(\boldsymbol{\rho})$ and where $\Re$ denotes the real part. It follows that for any choice of the functions f and g, Q has to be nonnegative. To find a necessary condition for $\hat{W}$ to represent a *bona fide* CSD matrix, let Q_{df} and Q_{xy} be the contributions to Q coming from the terms associated with W_{df} and W_{xy}, respectively, and observe that, because of Eq. (19), Q_{df} can be written as

$$Q_{df} = \sum_n \lambda_n\left(\left|\int \mathrm{d}^2\rho\Phi_n(\boldsymbol{\rho})f^*(\boldsymbol{\rho})\right|^2 + \left|\int \mathrm{d}^2\rho\Phi_n(\boldsymbol{\rho})g^*(\boldsymbol{\rho})\right|^2\right). \tag{22}$$

Further, on applying Parseval's theorem for the FT, the following equation holds for any n:

$$\int \mathrm{d}^2\rho\Phi_n(\boldsymbol{\rho})h^*(\boldsymbol{\rho}) = \int \mathrm{d}^2p\tilde{\Phi}_n(\boldsymbol{p})\tilde{h}^*(\boldsymbol{p}), \tag{23}$$

where $h=f,g$. Let us now recall that, according to Eq. (17), $\tilde{\Phi}_n$ is concentrated along a circle of radius $\kappa/(2\pi)$. Now, since f and g are arbitrary, we can always choose them such that $\tilde{f}=0$ and $\tilde{g}=0$ along the circle $p=\kappa/(2\pi)$. In this case, all terms in the sum on the r.h.s. of Eq. (22) vanish, implying that $Q_{df}=0$. As a consequence, on expressing f and g via their FTs, Q reduces to [see Eq. (13)]

$$Q = Q_{xy} = 2\,\Re\left\{\int\int \mathrm{d}^2p_1\mathrm{d}^2p_2\tilde{W}_{xy}(\boldsymbol{p}_1,-\boldsymbol{p}_2)\tilde{f}^*(\boldsymbol{p}_1)\tilde{g}(\boldsymbol{p}_2)\right\}. \quad (24)$$

Since $\tilde{f}$ and $\tilde{g}$ are required to vanish only along the circle of radius $\kappa/(2\pi)$, it is always possible to adjust them in the remaining parts of the $\boldsymbol{p}_1$and $\boldsymbol{p}_2$ planes in such a way that Q becomes negative, thereby implying that the CSD matrix will not be *bona fide* (i.e., it will not satisfy the nonnegative definiteness condition).

The latter statement can be proved by choosing $\tilde{f}(\boldsymbol{p})$ and $\tilde{g}(\boldsymbol{p})$ to be deltalike functions [remember Eq. (9)]. More precisely, we set $\tilde{f}(\boldsymbol{p})=a\,\delta(\boldsymbol{p}-\boldsymbol{p}_0)$ and $\tilde{g}(\boldsymbol{p})=b\,\delta(\boldsymbol{p}-\boldsymbol{p}_0')$, where a and b are arbitrary complex numbers, while $\boldsymbol{p}_0$ and $\boldsymbol{p}_0'$ denote two arbitrary points across the transverse plane, located outside the circle of radius $\kappa/2\pi$. With this choice Eq. (24) becomes

$$Q = Q_{xy} = 2\Re\{ab^*\tilde{W}_{xy}(\boldsymbol{p}_0,-\boldsymbol{p}_0')\}, \quad (25)$$

which, due to the arbitrariness of a and b, can always be made negative unless $W_{xy}(\boldsymbol{p}_0,-\boldsymbol{p}_0')$ vanishes. Since $\boldsymbol{p}_0$ and $\boldsymbol{p}_0'$ are arbitrary, we have proved that $\tilde{W}_{xy}(\boldsymbol{p}_1,-\boldsymbol{p}_2)$ has to vanish everywhere except at $p_1=p_2=\kappa/(2\pi)$, implying that

$$\tilde{W}_{xy}(\boldsymbol{p}_1,-\boldsymbol{p}_2) = \delta\left(p_1-\frac{\kappa}{2\pi}\right)\delta\left(p_2-\frac{\kappa}{2\pi}\right)F(\theta_1,\theta_2), \quad (26)$$

where F is a suitable function of the angular coordinates θ_1,θ_2 in the Fourier space.

Equation (26) specifies a necessary constraint on the mathematical structure of $\tilde{W}_{xy}$. To determine in which cases it is also a sufficient condition, we have to inquire about the choices of F leading to nonnegative Q when f and g are chosen at will. Notice that, in this case, $\tilde{f}$ and $\tilde{g}$ do not have to vanish along the circle of radius $\kappa/(2\pi)$. Observe now that using Eqs. (22), (23), and (17), Q_{df} can be expressed as

$$Q_{df} = \kappa^2\sum_n \lambda_n(|f_n|^2+|g_n|^2), \quad (27)$$

where

$$h_n = \frac{1}{2\pi}\int_0^{2\pi}\tilde{h}\left(\frac{\kappa}{2\pi},\theta\right)\exp(-\mathrm{i}n\theta)\mathrm{d}\theta, \quad (28)$$

with $h=f,g$ denoting the nth Fourier coefficient of the restriction, to the circle $p=\kappa/(2\pi)$, of the function $\tilde{h}(p,\theta)$.

As far as Q_{xy} is concerned, on inserting Eq. (26) into Eq. (24), and on taking Eq. (28) into account, we obtain, after some algebra, the expression

$$Q_{xy} = 2\kappa^2\,\Re\left\{\sum_{n,m} f_n^* g_m F_{nm}\right\}, \quad (29)$$

where the matrix elements F_{nm} are defined by

$$F_{nm} = \int_0^{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_2\mathrm{d}\theta_1}{4\pi^2}F(\theta_1,\theta_2)\exp(\mathrm{i}n\theta_1-\mathrm{i}m\theta_2). \quad (30)$$

Accordingly, the nonnegative definiteness condition for the function $F(\theta_1,\theta_2)$ transforms into

$$\sum_n \lambda_n(|f_n|^2+|g_n|^2)+2\sum_{n,m}\Re\{f_n^*g_mF_{nm}\}\geq 0, \quad (31)$$

for any choice of f_n,g_n. Equation (31) is the necessary and sufficient condition to determine, via Fourier inversion of Eq. (30), all possible types of functions $F(\theta_1,\theta_2)$ leading to genuine diffraction-free beams.

4. EXAMPLES

A. Shift-Invariant Case

Consider the case of $F_{nm}=a_n\delta_{nm}$, with $\{a_n\}$ being a set of (generally) complex numbers. Consequently, F is an angularly shift-invariant function; i.e., it depends on the difference $\theta_2-\theta_1$ only. In fact, its Fourier series is easily seen to be

$$F(\theta_1,\theta_2) = \sum_n a_n\exp[\mathrm{i}n(\theta_2-\theta_1)], \quad (32)$$

It then follows at once from Eqs. (26), (32), and (15) that

$$W_{xy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \sum_n a_n\Phi_n^*(\boldsymbol{\rho}_1)\Phi_n(\boldsymbol{\rho}_2), \quad (33)$$

and, by substitution into Eq. (31), the quadratic form reduces to

$$Q = \sum_n \lambda_n(|f_n|^2+|g_n|^2)+2\,\Re\{a_nf_n^*g_n\}. \quad (34)$$

Accordingly, the nonnegative definiteness condition on Q simply implies that the inequalities

$$|a_n| \leq \lambda_n \quad (35)$$

must be fulfilled for any n. It is worth noting that the present class of correlation matrices admits a simple and instructive physical interpretation. In fact, by using Eqs. (19) and (33), it is straightforward to show that $\hat{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ can be written as

$$\hat{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \sum_n \hat{\mathcal{P}}_n\Phi_n^*(\boldsymbol{\rho}_1)\Phi_n(\boldsymbol{\rho}_2), \quad (36)$$

where $\hat{\mathcal{P}}_n$ is the matrix

$$\hat{\mathcal{P}}_n = \begin{bmatrix}\lambda_n & a_n\\ a_n^* & \lambda_n\end{bmatrix}. \quad (37)$$

It is seen from Eqs. (36) and (37) that the degree of polarization will change from one point to another because of the different spatial structure of the functions $\Phi_n(\boldsymbol{\rho})$. Equation (37) allows the electromagnetic modal expansion [30] of $\hat{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ to be derived in a straightforward way. In fact, it is sufficient to diagonalize each matrix $\hat{\mathcal{P}}_n$. It is characterized by the eigenvalues $\mu_n^{(\pm)}=\lambda_n\pm|a_n|$, associated with the eigenvectors, say $\mathbf{U}_n^{(\pm)}$, whose (normalized) Jones representation is given by

$$\mathbf{U}_n^{(\pm)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm\exp(\mathrm{i}\varepsilon_n) \\ 1 \end{pmatrix}, \tag{38}$$

with ε_n denoting the phase of a_n. Accordingly, the Mercer's expansion of $\hat{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ reads

$$\hat{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \sum_n \sum_{\ell=\pm} \mu_n^{(\ell)} \boldsymbol{\Psi}_n^{(\ell)}(\boldsymbol{\rho}_1)[\boldsymbol{\Psi}_n^{(\ell)}(\boldsymbol{\rho}_2)]^{\dagger}, \tag{39}$$

where

$$\boldsymbol{\Psi}_n^{(\pm)}(\boldsymbol{\rho}) = \Phi_n(\boldsymbol{\rho})\mathbf{U}_n^{(\pm)}. \tag{40}$$

The modal decomposition in Eq. (39) gives a "natural" representation of $\hat{W}$ as a superposition of uncorrelated Bessel beams each of which has a suitable (uniform) polarization state. It is remarkable that the presented case constitutes one of the few examples (to our knowledge) of beams for which the electromagnetic modal expansion can be unfolded in exact, analytical terms.

As a simple but illuminating example, we apply these results to the vectorial generalization of DADDF beams. The diagonal elements of the correlation matrix $\hat{W}$ are given by Eq. (18). As far as the off-diagonal element is concerned, we shall limit ourselves to the case corresponding to setting $a_n = a$ for any n, with a being a complex parameter. This choice leads to $W_{xy} = aJ_0(\kappa|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|)$, and the nonnegativeness condition simply requires that $|a| \leqslant 1 - |\chi|$. In the Fourier domain, it is seen at once that the field turns out to be δ-correlated across the circle $p = \kappa/(2\pi)$. It is interesting to note that the present cross-spectral density matrix can always be written as a sum of two matrices, say $\hat{W}^{(p)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ and $\hat{W}^{(u)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$, where

$$\hat{W}^{(p)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = J_0(\kappa|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|)\begin{bmatrix} |a| & a \\ a^* & |a| \end{bmatrix}, \tag{41}$$

and $\hat{W}^{(u)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ is a diagonal matrix whose elements have the form

$$W_{\alpha\alpha}^{(u)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = (1 - |a|)J_0(\kappa|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|) + \chi J_0(\kappa|\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2|). \tag{42}$$

We then have the superposition of two uncorrelated beams. One of them, specified by $\hat{W}^{(p)}$, is completely polarized, while the other, $(\hat{W}^{(u)})$, is totally unpolarized. This type of decomposition is customary for polarization matrices [31] but is less familiar for CSD matrices [32].

B. Specular Case

As a second example, we consider beams with centrosymmetric correlation functions. The latter term implies the existence of a center of inversion—or even a whole manifold of such centers, for instance, a line or a plane—such that the optical fields are maximally correlated at pairs of points symmetric with respect to the center(s). If, for instance, the center of inversion is chosen to coincide with the origin, the fields are perfectly correlated at pairs of points specified by the radius vectors $\boldsymbol{\rho}$ and $-\boldsymbol{\rho}$ in the transverse plane of the beam. Specular beams were first introduced in [33] within the framework of scalar coherence theory. Electromagnetic specular solitons were also considered and their unusual polarization properties were examined in [34].

In the present context, we assume the angular correlation function $F(\theta_1,\theta_2)$ to be a *real* function of $\theta_1 + \theta_2$,

$$F(\theta_1,\theta_2) = G(\theta_1 + \theta_2), \tag{43}$$

where

$$G(\theta) = \sum_n \eta_n \exp(\mathrm{i}n\theta), \tag{44}$$

η_n being a complex parameter such that $\eta_{-n} = \eta_n^*$. To find the conditions under which the CSD matrix $\hat{W}$ so defined turns out to be nonnegative definite, we observe that the diagonal element of $\hat{W}$, in the Fourier space $(\boldsymbol{p}_1,\boldsymbol{p}_2)$, assumes the form

$$\tilde{W}_{\mathrm{df}}(\boldsymbol{p}_1,-\boldsymbol{p}_2) = \delta\left(p_1 - \frac{\kappa}{2\pi}\right)\delta\left(p_2 - \frac{\kappa}{2\pi}\right)F_d(\theta_2 - \theta_1), \tag{45}$$

where

$$F_d(\theta) = \sum_n \lambda_n \exp(\mathrm{i}n\theta). \tag{46}$$

Therefore, the matrix $\hat{\mathcal{W}}(\boldsymbol{p}_1,\boldsymbol{p}_2)$ introduced in Section 2 takes on the form

$$\hat{\mathcal{W}}(\boldsymbol{p}_1,\boldsymbol{p}_2) = \delta\left(p_1 - \frac{\kappa}{2\pi}\right)\delta\left(p_2 - \frac{\kappa}{2\pi}\right) \times \begin{bmatrix} F_d(\theta_2 - \theta_1) & G(\theta_2 + \theta_1) \\ G(\theta_2 + \theta_1) & F_d(\theta_2 - \theta_1) \end{bmatrix}. \tag{47}$$

On performing a counterclockwise $\pi/4$ rotation of the reference frame, the matrix assumes the diagonal form, say $\hat{\mathcal{W}}'(\boldsymbol{p}_1,\boldsymbol{p}_2)$, given by

$$\hat{\mathcal{W}}'(\boldsymbol{p}_1,\boldsymbol{p}_2) = \delta\left(p_1 - \frac{\kappa}{2\pi}\right)\delta\left(p_2 - \frac{\kappa}{2\pi}\right) \times \begin{bmatrix} K_+(\theta_1,\theta_2) & 0 \\ 0 & K_-(\theta_1,\theta_2) \end{bmatrix}, \tag{48}$$

where

$$K_{\pm}(\theta_1,\theta_2) = F_d(\theta_2 - \theta_1) \pm G(\theta_2 + \theta_1). \tag{49}$$

Note that the reality of G plays an essential role in the derivation.

We now see that the proof of the nonnegative definiteness of $\hat{W}$ reduces to finding the conditions under which the two *scalar* kernels $F_d(\theta_2 - \theta_1) \pm G(\theta_2 + \theta_1)$ are *bona fide*. The latter implies that the quantity

$$Q = \int\int \mathrm{d}\theta_1 \mathrm{d}\theta_2 [F_d(\theta_2 - \theta_1) \pm G(\theta_2 + \theta_1)]q^*(\theta_1)q(\theta_2) \tag{50}$$

must be nonnegative for any choice of $q(\theta)$. On writing $Q = Q_d \pm Q_a$, with

$$Q_d = \int\int \mathrm{d}\theta_1 \mathrm{d}\theta_2 F_d(\theta_2 - \theta_1) q^*(\theta_1) q(\theta_2),$$

$$Q_a = \int\int \mathrm{d}\theta_1 \mathrm{d}\theta_2 G(\theta_1 + \theta_2) q^*(\theta_1) q(\theta_2), \tag{51}$$

the nonnegative definiteness statement translates into

$$Q_d \geqslant |Q_a|, \tag{52}$$

for any q. It can be inferred from Eqs. (46) and (44) that

$$Q_d = \sum_n \lambda_n |q_n|^2,$$

$$Q_a = \sum_n \eta_n q^*_{-n} q_n, \tag{53}$$

where the asterisk denotes a complex conjugate and

$$q_n = \int_0^{\pi} \mathrm{d}\theta q(\theta) \exp(-\mathrm{i}n\theta) \tag{54}$$

is proportional to the nth Fourier coefficient of the function $q(\theta)$. Accordingly, $Q_d \pm Q_a$ assume the forms

$$Q_d \pm Q_a = (\lambda_0 \pm \eta_0)|q_0|^2 + \sum_{n=1}^{\infty} \lambda_n |q_n|^2 + \lambda_{-n}|q_{-n}|^2 \pm 2\,\Re[\eta_n q^*_{-n} q_n], \tag{55}$$

which turn out to be nonnegative for any choice of the set $\{q_n\}$, if and only if the following conditions are fulfilled:

$$|\eta_0| \leqslant \lambda_0,$$

$$|\eta_n| \leqslant \sqrt{\lambda_n \lambda_{-n}}, \quad n \geqslant 1. \tag{56}$$

It is also interesting to point out that in the present case, the complete modal expansion of the CSD tensor $\hat{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ can be found explicitly. To this end, we first solve the problem in the spatial frequency domain by considering the following integral equations:

$$\int_0^{2\pi} K_{\pm}(\theta_1, \theta_2) \Psi^{(\pm)}(\theta_2) \mathrm{d}\theta_2 = 2\pi \mu^{(\pm)} \Psi^{(\pm)}(\theta_1), \tag{57}$$

where $\mu^{(\pm)}$ denote the eigenvalues and $\Psi^{(\pm)}(\theta)$ the corresponding eigenfunctions. Recalling Eqs. (46) and (44) and substituting from Eq. (49) into Eq. (57), we arrive at the following linear equation set:

$$\lambda_{-n}\psi_n \pm \eta_n \psi_{-n} = \mu \psi_n, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{58}$$

where

$$\psi_n = \frac{1}{2\pi} \int_0^{2\pi} \Psi(\theta) \exp(-\mathrm{i}n\theta) \mathrm{d}\theta \tag{59}$$

denotes the nth Fourier coefficient of the eigenfunction $\Psi(\theta)$. The linear system in Eq. (58) actually leads to the following *uncoupled* linear systems:

$$(\lambda_0 + \gamma_0)\psi_0 = \mu \psi_0,$$

$$\left. \begin{aligned} \lambda_{-n}\psi_n + \gamma_n \psi_{-n} &= \mu \psi_n, \\ \lambda_n \psi_{-n} + \gamma_n^* \psi_n &= \mu \psi_{-n}, \end{aligned} \right\} n \geqslant 1, \tag{60}$$

where $\gamma_n = \pm \eta_n$ $(n = 0, 1, 2 \ldots)$. The eigenvalues of Eq. (60) can be found via elementary algebra. In particular, we have for $n = 0$ that

$$\mu_0^{(\pm)} = \lambda_0 \pm \eta_0, \tag{61}$$

while for $n \geqslant 1$ the eigenvalues are independent of the sign chosen for $K_{\pm}$ and are given by

$$\mu_{n,1}^{(\pm)} = \frac{\lambda_n + \lambda_{-n}}{2} + \sqrt{\left(\frac{\lambda_n - \lambda_{-n}}{2}\right)^2 + |\eta_n|^2}, \quad (n \geqslant 1),$$

$$\mu_{n,2}^{(\pm)} = \frac{\lambda_n + \lambda_{-n}}{2} - \sqrt{\left(\frac{\lambda_n - \lambda_{-n}}{2}\right)^2 + |\eta_n|^2}, \quad (n \geqslant 1), \tag{62}$$

which, due to the inequality in Eq. (56), turn out to be always nonnegative. As far as the corresponding eigenfunctions are concerned, it is easy to prove that

$$\Psi_0^{(\pm)} = 1, \tag{63}$$

while for $n \geqslant 1$ it turns out that

$$\Psi_{n,j}^{(\pm)} = \psi_{n,j}^{(\pm)} \exp(\mathrm{i}n\theta) + \psi_{-n,j}^{(\pm)} \exp(-\mathrm{i}n\theta), \tag{64}$$

where

$$\psi_{n,j}^{(\pm)} = \frac{\pm \eta_n}{\sqrt{(\mu_{n,j}^{(\pm)} - \lambda_{-n})^2 + |\eta_n|^2}},$$

$$\psi_{-n,j}^{(\pm)} = \frac{\mu_{n,j}^{(\pm)} - \lambda_{-n}}{\sqrt{(\mu_{n,j}^{(\pm)} - \lambda_{-n})^2 + |\eta_n|^2}}, \tag{65}$$

and $j = \{1, 2\}$.

Once the Fourier transforms of the eigenfunctions have been found in the rotated frame, the corresponding expressions in the coordinate space may be obtained by a Fourier inversion.

5. SUMMARY

In summary, we have presented a novel class of electromagnetic partially correlated diffraction-free beams. The diagonal elements of their CSD correspond to previously introduced scalar dark and antidark diffraction-free (DADDF) beams. However, the freedom to choose the off-diagonal elements of the CSD, whose functional form is subject only to the nonnegative definiteness requirement, results in a multitude of novel diffraction-free beams with very flexible—and controllable—polarization characteristics. The presented results are not restricted to DADDF beams; rather, our approach pertains to any beams whose CSD's diagonal elements are formed by superpositions of uncorrelated Bessel modes. Finally, we note that although ideal diffraction-free beams are not realizable in the laboratory, the apertured versions of the proposed beams can be easily implemented. The finite-size-aperture effects on DADDF beam propagation in free space were previously discussed in [11].

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