

Complex Gaussian representation of statistical pulses

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Abstract: We develop a general representation for ensembles of non-stationary random pulses in terms of statistically uncorrelated, time-delayed, frequency-shifted Gaussian pulses which are classical counterparts of coherent states of a quantum harmonic oscillator. We show that the two-time correlation function describing second-order statistics of the pulses can be expanded in terms of the complex Gaussian pulses. We also demonstrate how the novel formalism can be applied to describe recently introduced Gaussian Schell-model pulses and pulse trains generated by typical mode-locked lasers.

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OCIS codes: (030.0030) Coherence and statistical optics; (320.0320) Ultrafast optics; (320.5550) Pulses.

References and links

1. T. Brabec and F. Krausz, "Intense few-cycle laser fields: Frontiers of nonlinear optics," *Rev. Mod. Phys.* **72**, 545–591 (2000).
2. E. Wolf, *Introduction to the Theory of Coherence and Polarization* (Cambridge University Press, 2007).
3. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, 1995).
4. M. Bertolotti, A. Ferrari, and L. Sereda, "Coherence properties of nonstationary polychromatic light sources," *J. Opt. Soc. Am. B* **12**, 341–347 (1995).
5. L. Sereda, M. Bertolotti, and A. Ferrari, "Coherence properties of nonstationary light wave fields," *J. Opt. Soc. Am. B* **15**, 695–705 (1998).
6. S. A. Ponomarenko, G. P. Agrawal and E. Wolf, "Energy spectrum of a nonstationary ensemble of pulses," *Opt. Lett.* **29**, 394–396 (2004).
7. B. Davis, "Measurable coherence theory for statistically periodic fields," *Phys. Rev. A* **76**, 043843 (2007).
8. P. Pääkkönen, J. Turunen, P. Vahimaa, A. T. Friberg, and F. Wyrowski, "Partially coherent Gaussian pulses," *Opt. Commun.* **204**, 53–58 (2002).
9. H. Lajunen, J. Tervo, J. Turunen, P. Vahimaa, and F. Wyrowski, "Spectral coherence properties of temporarily modulated stationary light sources," *Opt. Express* **11**, 1894–1899 (2003).
10. P. Vahimaa and J. Turunen, "Independent-elementary-pulse representation for non-stationary fields," *Opt. Express* **14**, 5007–5012 (2006).
11. Q. Lin, L. Wang, and S. Zhu, "Partially coherent light pulse and its propagation," *Opt. Commun.* **219**, 65–70 (2003).
12. M. Brunel and S. Coëtlemec, "Fractional-order Fourier formulation of the propagation of partially coherent light pulses," *Opt. Commun.* **230**, 1–5 (2004).
13. R. W. Schoonover, B. J. Davis, and P. S. Carney, "The generalized Wolf shift for cyclostationary fields," *Opt. Express* **17**, 4705–4711 (2009).
14. G. Arfken, *Mathematical Methods for Physicists* (Academic Press, 1970).
15. Although the slowly-varying envelope approximation breaks down for few-cycle long femtosecond pulses, the decomposition into the envelope and carrier wave makes sense even in this case, see Ref. [19] for details.
16. S. A. Ponomarenko, J. J. Greffet, and E. Wolf, "Diffusion of partially coherent beams in turbulent media," *Opt. Commun.* **208**, 1–8 (2002).
17. E. Wolf, "New theory of partial coherence in space-frequency domain. Part I: Spectra and cross-spectra of steady-state sources," *J. Opt. Soc. Am.* **72**, 343–351 (1982).

1. P-representation of statistical pulses: introduction and preliminaries

Relentless recent progress in ultrafast optics [1] has motivated the quest for a better insight into statistical features of non-stationary sources generating ultrashort optical pulses. To this end, the extension of the usual optical coherence theory, which deals with statistically stationary light [2, 3], is required. The pioneering work in this direction [4, 5] was followed by the exploration into non-stationary spectra of generic statistical pulses [6] and the development of coherence theory of cyclostationary random pulses [7]. Concurrently, particular models of partially coherent non-stationary sources, notably Gaussian Schell-model one, were introduced and the corresponding optical fields were explored [8–10]. The evolution of partially coherent pulses in linear dispersive media was also examined [11, 12] and the generalization of correlation-induced spectral changes to non-stationary random pulses was presented in Ref. [13].

The purpose of this work is to formulate a statistical theory of random pulses in the language that is sufficiently flexible to describe a variety of partially coherent pulse models on the one hand, and on the other hand, establishes a clear link with experimentally realizable ultrashort pulses. To this end, we represent each statistical pulse as a linear superposition of uncorrelated, time-delayed, frequency-shifted Gaussian pulses—which can be routinely produced in the laboratory by standard lasers—with a statistical distribution of emission times and carrier frequency shifts. The complex Gaussian pulses are classical analogues of coherent states of a quantum harmonic oscillator. By analogy with the Glauber-Sudarshan \mathcal{P} -representation in quantum optics [3], we can then express the second-order two-time correlation function of any statistical pulse as an integral over an over-complete non-orthogonal set of complex Gaussian pulses. We then discuss the application of the advanced representation to several particular cases of practical interest.

We start by considering a time-delayed by t_s Gaussian pulse with the carrier frequency shifted to ω_s ; the pulse has the temporal profile

$$\psi(t; t_s, \omega_s) = A \exp \left[-\frac{(t - t_s)^2}{2t_*^2} \right] e^{i\omega_s t}, \quad (1)$$

where A and t_* are a real amplitude and width of the pulse. Transforming to dimensionless variables, $T = t/t_*$, $T_s = t_s/t_*$, and $\Omega_s = \omega_s t_*$ —which we are going to use hereafter unless we indicate otherwise—we obtain, after elementary algebra, the following expression

$$\psi_\alpha(T) = \frac{e^{-(\text{Im}\alpha)^2}}{\pi^{1/4}} \exp \left[-\frac{(T - \sqrt{2}\alpha)^2}{2} \right]. \quad (2)$$

Here the complex displacement conveniently combines time delay and frequency shift viz.,

$$\alpha = \frac{1}{\sqrt{2}}(T_s + i\Omega_s). \quad (3)$$

In Eq. (2) we chose the amplitude A such that the pulse profile function is normalized to unity:

$$\int_{-\infty}^{\infty} dT |\psi_\alpha(T)|^2 = 1. \quad (4)$$

Let us now look at an unnormalized coherent state of the quantum harmonic oscillator [3]

$$|\alpha\rangle = \mathcal{A} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (5)$$

It follows from Eq. (5) that the coordinate-representation wave function of the coherent state is given by

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \mathcal{A} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle x|n\rangle, \quad (6)$$

where the number state $|n\rangle$ can be expressed in the coordinate representation as

$$\langle x|n\rangle = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} H_n(x) e^{-x^2/2}. \quad (7)$$

Performing summation over n on the r.h.s of Eq. (6) with the aid of Eq. (7) and the generating function for Hermite polynomials $H_n(x)$ in the form [14]

$$e^{2sx-s^2} = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(x), \quad (8)$$

we arrive at the final expression for the normalized coherent state wave function in the coordinate representation as

$$\psi_\alpha(x) = \frac{e^{-(\text{Im}\alpha)^2}}{\pi^{1/4}} \exp\left[-\frac{(x - \sqrt{2}\alpha)^2}{2}\right]. \quad (9)$$

Equation (9) is identical with Eq. (2) apart from the notation. Thus, the (normalized) complex Gaussian pulses have the same profiles as the coherent states.

As a consequence of the outlined mathematical equivalence, the complex Gaussian pulses form a complete set such that

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = 1. \quad (10)$$

Alternatively, by introducing time bra- and ket-vectors, $\langle T|$ and $|T\rangle$, we can re-write the completeness relation explicitly in the temporal representation as

$$\int d^2\alpha \psi_\alpha(T) \psi_\alpha^*(T') = \delta(T - T'), \quad (11)$$

where we denoted $\psi_\alpha(T) = \langle T|\alpha\rangle$.

2. P-representation of statistical pulses: general formalism

Consider an ensemble of random pulses $\{E(T)\}$. Hereafter, we find it convenient to decompose the electric field $E(T)$ into a (usually) slowly-varying envelope $U(T)$ and a carrier wave [15] such that

$$E(T) = U(T) e^{-i\Omega_c T}, \quad (12)$$

where Ω_c is a deterministic carrier frequency of the pulse. The second-order statistical properties of the ensemble $\{U(T)\}$ are specified by the cross-correlation function

$$\Gamma(T_1, T_2) = \langle U^*(T_1) U(T_2) \rangle, \quad (13)$$

where the angle brackets denote ensemble averaging. Similarly to the spatial case [16], we can introduce a statistical operator $\hat{\Gamma}$ such that its matrix elements correspond to the two-time correlation function,

$$\Gamma(T_1, T_2) \equiv \langle T_2 | \hat{\Gamma} | T_1 \rangle. \quad (14)$$

Following the quantum optical development, we can represent the statistical operator in a diagonal form as

$$\hat{\Gamma} = \int d^2\alpha \mathcal{P}(\alpha) |\alpha\rangle\langle\alpha|. \quad (15)$$

Equation (15) is a classical analogue of the well-known Glauber-Sudarshan \mathcal{P} -representation of the quantum density operator [3]; it can be written in the matrix form as

$$\langle T_2 | \hat{\Gamma} | T_1 \rangle = \int d^2\alpha \mathcal{P}(\alpha) \langle T_2 | \alpha \rangle \langle \alpha | T_1 \rangle, \quad (16)$$

implying in more “classical” notations that

$$\Gamma(T_1, T_2) = \int d^2\alpha \mathcal{P}(\alpha) \psi_\alpha^*(T_1) \psi_\alpha(T_2). \quad (17)$$

Notice that one can formally invert Eq. (17) to determine the classical P-distribution in terms of the two-time correlation function. With this purpose, we can again borrow the known expression for $\mathcal{P}(\alpha)$ in the matrix form [3]

$$\mathcal{P}(\alpha) = \frac{1}{\pi^2} \int d^2\beta e^{|\alpha|^2 + |\beta|^2} \langle -\beta | \Gamma | \beta \rangle e^{\beta^* \alpha - \alpha^* \beta}, \quad (18)$$

and expand the rhs in terms of complete sets of time ket-vectors to yield

$$\begin{aligned} \mathcal{P}(\alpha) &= \frac{e^{|\alpha|^2}}{\pi^2} \int d^2\beta e^{|\beta|^2} e^{\beta^* \alpha - \alpha^* \beta} \\ &\times \int_{-\infty}^{\infty} dT_1 \int_{-\infty}^{\infty} dT_2 \Gamma(T_1, T_2) \psi_{-\beta}^*(T_1) \psi_\beta(T_2). \end{aligned} \quad (19)$$

In principle, Eqs. (17) and (19) solve the problem of finding the appropriate complex Gaussian representation for any statistical pulse. In practice, of course, the integrals in Eq. (19) can fail to converge in the space of ordinary functions, making the P-representation cumbersome in the case.

An alternative—yet equivalent—statistical representation of random pulses is arrived at by examining an expansion of a statistical ensemble member $U(T)$ in terms of the complex Gaussian pulses with random amplitudes $c(\alpha)$ as

$$U(T) = \int d^2\alpha c(\alpha) \psi_\alpha(T). \quad (20)$$

We conclude with the help of Eq. (13) that for the expansion (Eq. (20)) to be compatible with the P-representation (Eq. (17)), $\{c(\alpha)\}$ must be uncorrelated, obeying

$$\langle c^*(\alpha') c(\alpha) \rangle = \mathcal{P}(\alpha) \delta(\alpha - \alpha'). \quad (21)$$

The just derived stochastic expansion can serve as a good starting point for synthesizing new partially coherent pulses from complex Gaussian ones.

It is instructive to compare the developed representation with a coherent-mode decomposition of optical coherence theory, originally formulated for spatial fields [17]. According to the latter, the cross-correlation function of the pulse can be expanded into a Mercer-type series as

$$\Gamma(T_1, T_2) = \sum_n \lambda_n \phi_n^*(T_1) \phi_n(T_2), \quad (22)$$

where the modes $\phi_n(T)$ form a complete orthonormal set such that

$$\int_{-\infty}^{\infty} dT \phi_m^*(T) \phi_n(T) = \delta_{nm}. \quad (23)$$

Each mode and the corresponding eigenvalue λ_n are determined by solving the linear integral equation in the form

$$\int_{-\infty}^{\infty} dT_1 \Gamma(T_1, T_2) \phi_n(T_1) = \lambda_n \phi_n(T_2). \quad (24)$$

Equivalently, each ensemble member $U(T)$ can be represented using the Karhunen-Loève expansion

$$U(T) = \sum_n a_n \phi_n(T), \quad (25)$$

where the stochastic amplitudes are uncorrelated and normalized such that

$$\langle a_m^* a_n \rangle = \lambda_n \delta_{nm}. \quad (26)$$

Notice that the modes of the coherent-mode theory can have arbitrary temporal profiles, depending on particulars of the pulse statistics. The modes are determined by solving the integral equation (Eq. (24)), which can be a formidable mathematical task. In contrast, the advanced P-representation is always formulated in terms of complex Gaussian pulses. Not only are the latter mathematically well-behaved and physically realizable, but they also remain shape-invariant on propagation through linear temporal elements—including time-lenses—and linear dispersive media. Hence, the advanced P-representation is expected to be superior to the coherent-mode approach whenever P-distributions of pulses turn out to be well-behaved ordinary functions.

It is also instructive to compare the complex Gaussian representation with the elementary-pulse-representation introduced in Ref. [10]. While the former can be used either to concoct novel sources or to represent the sources with given cross-correlation functions using Eqs. (17) and (19), the elementary pulse envelopes do not, in general, form a complete set and hence do not allow for a general cross-correlation function representation.

3. Examples and discussion

As the first example, we examine the P-representation of a recently introduced [8, 9] nonstationary Gaussian Schell-model (GSM) source with the cross-correlation function

$$\Gamma(T_1, T_2) = I_0 \exp \left[-\frac{T_1^2 + T_2^2}{2\sigma_p^2} \right] \exp \left[-\frac{(T_1 - T_2)^2}{2\sigma_c^2} \right], \quad (27)$$

where we introduced the dimensionless pulse width and coherence time: $\sigma_p = t_p/t_*$ and $\sigma_c = t_c/t_*$. Substituting from Eq. (27) into Eq. (19), we obtain, after lengthy but straightforward algebra, the P-distribution of the GSM pulsed source in the general form

$$\begin{aligned} \mathcal{P}(T_s, \Omega_s) &= \frac{2I_0}{\sqrt{\pi^2(1 - 1/\sigma_p^2)(2/\sigma_c^2 + 1/\sigma_p^2 - 1)}} \\ &\times \exp \left[-\frac{T_s^2}{\sigma_p^2(1 - 1/\sigma_p^2)} - \frac{\Omega_s^2}{2/\sigma_c^2 + 1/\sigma_p^2 - 1} \right]. \end{aligned} \quad (28)$$

Here we expressed the answer in physical variables T_s and Ω_s related to α by Eq. (5). As is seen from Eq. (28), the scaling factor t_* —the width of a complex Gaussian pulse—serves as an additional degree of freedom in choosing the most adequate P-distribution for a given source model. In this case, the choice $\sigma_p \rightarrow 1$, implying that $t_* \rightarrow t_p$ and $\sigma_c = t_c/t_p$, leads to the simplest and most physically transparent representation

$$\mathcal{P}(T_s, \Omega_s) = \frac{I_0 \sigma_c}{\sqrt{\pi}} \delta(T_s) e^{-\sigma_c^2 \Omega_s^2 / 2}. \quad (29)$$

In physics terms, Eq. (29) implies that a nonstationary GSM source of width t_p can be represented by a superposition of statistically uncorrelated Gaussian pulses of the same width, with no time delay and a Gaussian distribution of frequency shifts; the coherence time of the source determines the distribution width. We note in passing that a somewhat similar representation to Eq. (29) can also be synthesized using the elementary-pulse-representation technique of Ref. [10].

Another instructive application of the new formalism lies in the area of partially coherent source modeling. In particular, novel partially coherent sources can be straightforwardly synthesized by mixing a countable number of uncorrelated complex Gaussian pulses such that

$$\mathcal{P}(\alpha) = \sum_n w_n \delta(\alpha - \alpha_n), \quad (30)$$

where $w_n \geq 0$ specifies the energy carried by the n th Gaussian pulse. It then follows from Eqs. (17) and (30) that in the dimensional variables, the two-time cross-correlation function takes the form

$$\Gamma(t_1, t_2) = \sum_n w_n \psi_{\alpha_n}^*(t_1) \psi_{\alpha_n}(t_2). \quad (31)$$

Let us specialize to the case of $\alpha_n = n(t_0/t_* + i\omega_0 t_*)/\sqrt{2}$, where t_0 and t_* characterize the individual Gaussian pulse peak time and width, respectively, and $-N \leq n \leq N$. The averaged intensity profile of the resulting random pulse train takes the form

$$I(t) = \sum_{n=-N}^N w_n |\psi_{\alpha_n}(t)|^2 = \frac{1}{\sqrt{\pi} t_*} \sum_{n=-N}^N w_n \exp\left[-\frac{(t - nt_0)^2}{t_*^2}\right]. \quad (32)$$

Provided $N \gg 1$ and $t_* \ll t_0$, Eq. (32) describes rather well the intensity of a random train of realistic ultrashort mode-locked pulses with the individual pulses centered at the integer multiples of t_0 , having a width of t_* ; further, if $w_n = w_0 = \text{const}$ and $t_* = t_0/N$, we have a periodic train of identical mode-locked pulses generated in a cavity with a round-trip transit time of t_0 [18, 19].

We note in passing that in the fully coherent case, the pulse field can be expressed by Eq. (20) with deterministic amplitudes $\{c(\alpha)\}$. In particular, considering

$$c(\alpha) = \sum_n c_n \delta(\alpha - \alpha_n), \quad (33)$$

with $\alpha_n = n(t_0/t_* + i\omega_0 t_*)/\sqrt{2}$, we obtain a train of coherent Gaussian pulses with the field profile

$$U(t) = \frac{1}{\sqrt{\pi^{1/2} t_*}} \sum_{n=-N}^N c_n e^{i n \omega_0 t_*} \exp\left[-\frac{(t - nt_0)^2}{2 t_*^2}\right]. \quad (34)$$

Equation (34) represents an ideal train of identical coherent Gaussian pulses provided that $c_n = c_0 = \text{const}$, $\omega_0 = 2\pi/t_0$ and $t_* = t_0/N$.

To summarize, we presented a novel formalism for describing statistical properties of ultrashort random pulses. The proposed approach is based on the expansion–diagonal representation akin to the Glauber-Sudarshan P -representation of quantum optics—of the cross-correlation function of any statistical pulse in terms of complex Gaussian pulses with the appropriately distributed emission times and carrier frequencies. We showed how the complex Gaussian representation can describe statistical features of Gaussian Schell-model pulses and the output of realistic mode-locked lasers. The new representation is anticipated to find applications in ultrafast optics and temporal imaging with ultrashort pulses.