Effective spatial and angular correlations in beams of any state of spatial coherence and an associated phase-space product

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We study effective spatial and angular correlations in beams of any state of spatial coherence, and we introduce a phase-space product, Q, which takes these correlations into account. This phase-space product is shown to reduce to the conventional beam-quality factor M^2 when the beam is spatially fully coherent. We also determine the lower bound for the value of Q and demonstrate that it is attained for all Gaussian Schell-model beams.

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The characterization of beams of an arbitrary profile and any state of spatial coherence has been an important issue in the design of laser devices. The current interest in this subject stems from the discovery that on propagation through the atmosphere, a partially coherent beam is less affected by turbulence than is a fully coherent beam.[1] This interest has been further stimulated by the advances in development of partially coherent light sources such as diode lasers and, more recently, superluminescent diodes.[2] In this context, it is desirable to generate partially coherent beams of a relatively small effective cross section of the beam in the source plane and a small effective angular divergence of the beam. The question then arises as to how one can introduce a suitable phase-space measure for such beams.

To date, a number of papers have been published that deal with the second-order moments of the spatial and angular intensity distributions of partially coherent beams (see, for example, Refs. 3–13). A possible choice for the appropriate phase-space product of a beam, which involves the second-order moments of the Wigner distribution function, has been introduced and measured.[4] Later, the term M^2 quality factor for this product was coined by Siegman.[14] However, the beam cross section in the source plane has been generally introduced in terms of a second-order moment of the intensity distribution.[3–13] Defined this way, the cross section is independent of the coherence properties of the source. In order to examine the influence of the state of coherence of the source on the value of the phase-space product of the beam generated by the source, one has to define the effective cross section and the effective angular spread so that these quantities contain information about both the intensity of the source *and* its coherence properties.

In this Letter, we study the spatial correlations of a partially coherent source and the angular correlations of the beam generated by the source, and we introduce a *correlation-based* measure Q of the phase-space characteristics of beams of any state of coherence.

Let us consider the field generated by a planar, secondary, statistically stationary source of any state of coherence, situated in the plane $z = 0$ and radiating into the half-space $z > 0$. Let $W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ denote the cross-spectral density of the light at frequency ω (Ref. 15, Sect. 4.3.2) at a pair of points with position vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ in the source plane (see Fig. 1). $W^{(0)}$ represents the spatial correlations at frequency ω, which exist at pairs of points across the source. We define the normalized width δ_W of $W^{(0)}$ by the expression

$$\delta_W^2 = \frac{\int d^2\rho_1 \int d^2\rho_2 (\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2 |W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)|^2}{\int d^2\rho_1 \int d^2\rho_2 |W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)|^2}, \quad (1)$$

where the integrations extend over the source.

Next, we introduce the angular correlation function $\mathcal{A}(\mathbf{s}_1, \mathbf{s}_2, \omega)$, which is a measure of the degree of correlation of the field in the far zone, at points P_1 and P_2, in the directions specified by unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively (see Fig. 1). We define the normalized

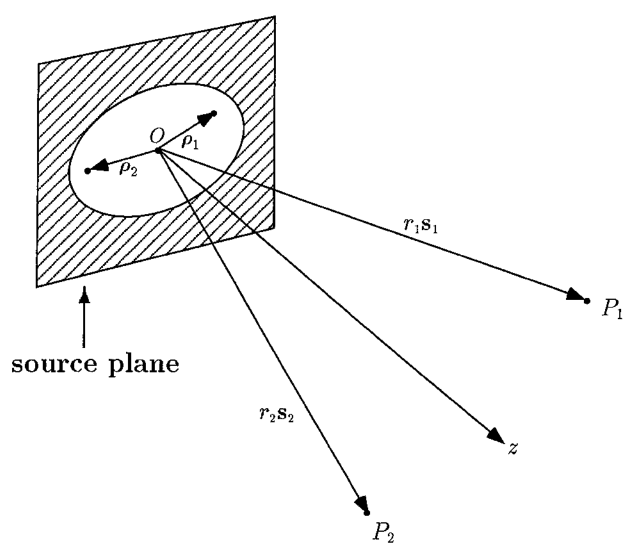

Fig. 1. Illustration of the notation. The point O is an origin within the region in the plane $z = 0$, occupied by a partially coherent, statistically stationary, planar, secondary source. P_1 and P_2 are two points in the far zone of the source, situated in the plane $z = 0$. $\overline{OP_1} = r_1\mathbf{s}_1$, $\overline{OP_2} = r_2\mathbf{s}_2$, $(\mathbf{s}_1^2 = \mathbf{s}_2^2 = 1)$.

width $\delta_{\mathcal{A}}$ of the angular correlation function by the formula

$$\delta_{\mathcal{A}}^2 = \frac{\int \mathrm{d}^2 s_{1\perp} \int \mathrm{d}^2 s_{2\perp} (\mathbf{s}_{1\perp} - \mathbf{s}_{2\perp})^2 |\mathcal{A}(\mathbf{s}_1, \mathbf{s}_2, \omega)|^2}{\int \mathrm{d}^2 s_{1\perp} \int \mathrm{d}^2 s_{2\perp} |\mathcal{A}(\mathbf{s}_1, \mathbf{s}_2, \omega)|^2} . \quad (2)$$

In this expression, $\mathbf{s}_{1\perp}$ and $\mathbf{s}_{2\perp}$ are the projections, considered as two-dimensional vectors, of the unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ onto the source plane $z = 0$. The integrations on the right-hand side of Eq. (2) are formally taken over the entire $\mathbf{s}_{1\perp}$ and $\mathbf{s}_{2\perp}$ planes, although in practice only those $\mathbf{s}$ directions in which the far field is appreciable will contribute. Further, the angular correlation function $\mathcal{A}$ of a radiated field can be shown to be related to the cross-spectral density $W^{(0)}$ in the source plane by the expression (Ref. 15, Sect. 5.6.3)

$$\mathcal{A}(\mathbf{s}_1, \mathbf{s}_2, \omega) = k^4 \tilde{W}^{(0)}(-k\mathbf{s}_{1\perp}, k\mathbf{s}_{2\perp}, \omega) , \quad (3)$$

where $k = \omega/c = 2\pi/\lambda$, λ being the wavelength of light at frequency ω, and $\tilde{W}^{(0)}(\mathbf{f}_1, \mathbf{f}_2, \omega)$ is the four-dimensional Fourier transform of $W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$, i.e.,

$$\tilde{W}^{(0)}(\mathbf{f}', \mathbf{f}, \omega) = \iint \frac{\mathrm{d}^2\rho \mathrm{d}^2\rho'}{(2\pi)^2} W^{(0)}(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega) \times \exp[-i(\mathbf{f}\boldsymbol{\rho} + \mathbf{f}'\boldsymbol{\rho}')] . \quad (4)$$

We now define a new phase-space product for beams of any state of coherence by the expression

$$Q = \left(\frac{\pi}{\lambda}\right) \delta_W \delta_{\mathcal{A}} . \quad (5)$$

We will show that this correlation-based phase-space product can be regarded as a generalization to beams of arbitrary state of coherence of the well-known M^2 factor. We will demonstrate that in the limiting case of a fully coherent beam, the Q factor indeed reduces to the M^2 factor of Siegman. To show this, let us recall that if the source and, consequently, the radiated field are spatially fully coherent at frequency ω, the cross-spectral density factorizes; i.e., it is of the form (Ref. 15, Sect. 4.5.2)

$$W^{(\mathrm{coh})}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = V^*(\boldsymbol{\rho}_1, \omega) V(\boldsymbol{\rho}_2, \omega) , \quad (6)$$

and the "diagonal element" of W is then just the optical intensity $I(\boldsymbol{\rho}, \omega) = |V(\boldsymbol{\rho}, \omega)|^2$. Therefore, in the fully coherent case [writing now $(\delta_W^2)_{\mathrm{coh}}$ for δ_W^2 and omitting the ω dependence for brevity], Eq. (1) becomes

$$(\delta_W^2)_{\mathrm{coh}} = \frac{\int \mathrm{d}^2\rho_1 \int \mathrm{d}^2\rho_2 (\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2 I(\boldsymbol{\rho}_1) I(\boldsymbol{\rho}_2)}{\int \mathrm{d}^2\rho_1 \int \mathrm{d}^2\rho_2 I(\boldsymbol{\rho}_1) I(\boldsymbol{\rho}_2)} . \quad (7)$$

The first- and the second-order moments of the optical intensity are defined by the formulas

$$\langle \rho_\alpha \rangle = \frac{\int \mathrm{d}^2\rho \rho_\alpha I(\boldsymbol{\rho})}{\int \mathrm{d}^2\rho I(\boldsymbol{\rho})} , \quad (8)$$

$$\langle \boldsymbol{\rho}^2 \rangle = \frac{\int \mathrm{d}^2\rho \boldsymbol{\rho}^2 I(\boldsymbol{\rho})}{\int \mathrm{d}^2\rho I(\boldsymbol{\rho})} . \quad (9)$$

The subscript α in Eq. (8), which takes on values 1 and 2, labels the Cartesian components of the two-dimensional vector $\boldsymbol{\rho}$ in the source plane. It follows from Eq. (7) that

$$(\delta_W^2)_{\mathrm{coh}} = (\langle \boldsymbol{\rho}^2 \rangle + \langle \boldsymbol{\rho}^2 \rangle - 2\langle \boldsymbol{\rho} \rangle^2) , \quad (10)$$

so that

$$(\delta_W^2)_{\mathrm{coh}} = 2\sigma_I^2 , \quad (11)$$

where σ_I^2 is the variance of the intensity. Thus in the coherent limit, δ_W^2 is just twice the variance of the source intensity.

Further, we introduce the radiant intensity $J(\mathbf{s}, \omega)$ of the field produced by the source, which is known to be related to the angular correlation function by the expression (Ref. 15, Sect. 5.2.1)

$$J(\mathbf{s}, \omega) = \left(\frac{2\pi}{k}\right)^2 \mathcal{A}(\mathbf{s}, \mathbf{s}, \omega) \cos^2\theta , \quad (12)$$

where θ is the angle that the unit vector $\mathbf{s}$ makes with the z axis. For a beam, the factor $\cos^2\theta$ can be approximated by unity. Also, for a fully coherent beam, the angular correlation function $\mathcal{A}(\mathbf{s}_1, \mathbf{s}_2, \omega)$, just as the cross-spectral density, factorizes in terms of the amplitudes $a(\mathbf{s}, \omega)$ in the angular spectrum representation of the field, viz,

$$\mathcal{A}^{(\mathrm{coh})}(\mathbf{s}_1, \mathbf{s}_2, \omega) = a^*(\mathbf{s}_1, \omega) a(\mathbf{s}_2, \omega) . \quad (13)$$

It follows from Eqs. (12), (13), and (2) that in the fully coherent case the width of the angular correlation function is given by the formula

$$(\delta_{\mathcal{A}}^2)_{\mathrm{coh}} = \frac{\int \mathrm{d}^2 s_{1\perp} \int \mathrm{d}^2 s_{2\perp} (\mathbf{s}_{1\perp} - \mathbf{s}_{2\perp})^2 J(\mathbf{s}_1) J(\mathbf{s}_2)}{\int \mathrm{d}^2 s_{1\perp} \int \mathrm{d}^2 s_{2\perp} J(\mathbf{s}_1) J(\mathbf{s}_2)} . \quad (14)$$

By an argument similar to the one employed in the derivation of Eq. (11), one readily finds that

$$(\delta_{\mathcal{A}}^2)_{\mathrm{coh}} = 2(\langle \mathbf{s}_\perp^2 \rangle - \langle \mathbf{s}_\perp \rangle^2) = 2\sigma_J^2 , \quad (15)$$

where σ_J^2 is the variance of the radiant intensity. Hence, in the coherent limit, $\delta_{\mathcal{A}}$ is proportional to the angular spread of the beam. On combining Eqs. (11) and Eq. (15), we see that in the coherent limit, Eq. (5) reduces to

$$Q_{\mathrm{coh}} = \left(\frac{2\pi}{\lambda}\right) \sigma_I \sigma_J , \quad (16)$$

which is just the M^2 factor of Siegman.[5]

Next, we determine the lower bound for the value of Q. For this purpose, we consider the inequality

$$\frac{\int \mathrm{d}^2\rho_1 \int \mathrm{d}^2\rho_2 (\mathbf{g}^* \cdot \mathbf{g}) |W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)|^2}{\int \mathrm{d}^2\rho_1 \int \mathrm{d}^2\rho_2 |W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)|^2} \geq 0\,, \tag{17}$$

where

$$\mathbf{g} = \sum_{j=1}^{2} (\boldsymbol{\rho}_j + \alpha \nabla_j) W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\,. \tag{18}$$

Here the subscript j, which takes on values 1 and 2, labels a pair of points entering the definition (1), and α is an arbitrary real number. Now according to Eq. (3), the angular correlation function and the cross-spectral density in the source plane form a Fourier transform pair. Hence, on utilizing the properties of pairs of Fourier transforms along the lines of Ref. 12, we may convert inequality (17) into the inequality

$$\delta_W^2 + 4\alpha + \alpha^2 k^2 \delta_{\mathcal{A}}^2 \geq 0\,. \tag{19}$$

Equation (5) and inequality (19) imply that

$$Q \geq 1\,. \tag{20}$$

This inequality provides a lower bound for Q. It then follows from inequalities (17) and (19) that in order for a source to generate a beam with the smallest phase-space product Q, one must have $\mathbf{g} \equiv 0$. This condition, together with the definition (18) of $\mathbf{g}$, results in the following differential equation for the unknown cross-spectral density of the source $W_{\min}^{(0)}$:

$$(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) W_{\min}^{(0)} + \alpha_0 (\nabla_1 - \nabla_2) W_{\min}^{(0)} = 0\,. \tag{21}$$

The solution to Eq. (21) is

$$W_{\min}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = A \exp[-\gamma(\boldsymbol{\rho}_1^2 + \boldsymbol{\rho}_2^2)] \exp(-\beta \boldsymbol{\rho}_1 \boldsymbol{\rho}_2)\,, \tag{22}$$

where A, β, and γ are positive real constants. On substituting from Eq. (22) into Eq. (21), one obtains an algebraic relation among the constants α_0, β, and γ. Expression (22) indicates that the general solution of Eq. (21) is a Gaussian function with two independent parameters. We can recast expression (22) into a more familiar form,

$$W_{\min}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = A \exp[-(\boldsymbol{\rho}_1^2 + \boldsymbol{\rho}_2^2)/4\sigma_S^2] \times \exp[-(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2/2\sigma_g^2]\,, \tag{23}$$

which is the cross-spectral density of a Gaussian Schell-model source (Ref. 15, Sect. 5.3.2), with σ_S being the width of the intensity distribution and σ_g being the spatial coherence length. Thus we have shown that *not only the fully coherent Gaussian beam but also any partially coherent Gaussian Schell-model beam minimizes the generalized phase-space product Q*.

Finally, let us compare the present approach with some others discussed in the literature. We have defined the effective size and the effective angular width in terms of the appropriate *correlations*, thus taking into account coherence properties of the source at a *local* level, via the spectral degree of coherence. Bastiaans,[6–8] on the other hand, took account of the coherence properties of the source by using quite a different approach. He utilized the same definitions of the source size and the beam spread as those used in the coherent case but introduced an overall degree of coherence of the source, which amounts to incorporating its coherence properties at a *global* level. He then derived a series of inequalities involving the phase-space product as well as the overall degree of coherence. However, there appears to be no physical principle indicating which is the most appropriate measure of the overall degree of coherence. In fact, different choices of such measure have led Bastiaans to different results for the beams with minimum phase-space products.[7,8]

In summary, we have considered the spatial and the angular correlations in partially coherent beams as possible choices for describing the phase-space properties of such beams. We have then introduced a correlation-based phase-space product Q, and we have shown that it reduces to the usual M^2 quality factor in the coherent limit. We have also demonstrated that the phase-space product Q attains minimum for any Gaussian Shell-model beam.

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