# Correlations in open quantum systems and associated uncertainty relations

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We study how correlations in a state of an open quantum system affect the intrinsic uncertainties of the expectation values of an arbitrary pair of noncommuting observables. We show that for such observables, there exist Heisenberg-type uncertainty relations that take fully into account correlations in the state of the system. If the quantum system is in a pure state, such uncertainty relations reduce to the conventional one. We obtain an equation for the density operator of a general state that minimizes the new uncertainty relations, and demonstrate that in the important case of coordinate and momentum operators, the minimum-uncertainty states are displaced squeezed thermal states.

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## I. INTRODUCTION

The conventional uncertainty relation (UR) is a cornerstone of the modern quantum theory of measurement [1], and consequently, is treated in most quantum mechanics textbooks (see, for example, Ref. [2]). However, contrary to the widespread view, the usual ''textbook'' UR pertains to separate rather than joint measurements [3] of observables $\hat{A}$ *or* $\hat{B}$ on a system in a state $|\psi\rangle$. In essence, the uncertainty relation indicates that due to intrinsic indeterminacy of the quantum state, the product of the variances of two noncommuting observables, $\hat{A}$ and $\hat{B}$ cannot be less than a certain value, which is expressed mathematically as [2]:

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geqslant \tfrac{1}{4}|\langle\psi|[\hat{A},\hat{B}]_-|\psi\rangle|^2. \tag{1}$$

Here, $[,]_-$ denotes the commutator of a pair of operators, $[\hat{A},\hat{B}]_- \equiv \hat{A}\hat{B}-\hat{B}\hat{A}$, and $\langle(\Delta\hat{A})^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$ is the variance of the operator $\hat{A}$. It should be noted that the inequality (1) is formulated for a *pure* state. Uncertainty relations of this type were extensively studied as early as the 1930's [4]. There has also been a considerable interest in uncertainty relations associated with joint measurements of noncommuting observables [5–8] as well as in the generalized parameter-based UR's that do not explicitly depend on the expectation value of the commutator [9–11]. Further, a generalization of the Heisenberg-type UR (1) to open quantum systems was carried out [12,13], and the nature of the states that minimize such a generalized UR was examined. These studies have shown that, at least when the observables $\hat{A}$ and $\hat{B}$ are the coordinate and the momentum, the most general minimum-uncertainty state *must* be a pure state [12].

Since the knowledge of the minimum-uncertainty states is of importance for devising high-precision quantum measurement schemes, the relation of the nature of such states, i.e., the degree of their purity, to correlations in the state of the measured system, evidently warrants further investigation. Such correlations are described by off-diagonal elements, which are sometimes referred to as quantum coherences, of the density operator of the system. We stress that off-diagonal elements of the density operator are important not only in the state preparation for quantum measurements, but they also play a prominent role in the consideration of such a fundamental issue as decoherence of an open quantum system [14], as well as in the realization of lasing without inversion [15], and in studies of efficiency of quantum teleportation [16]. To our knowledge, however, research aiming at elucidating the influence of correlations in a mixed state on the structure of the associated UR, has so far been limited to the treatment of *only* one pair of observables, namely coordinate and momentum [17].

In the present paper, we consider separate measurements of either of the two *arbitrary* noncommuting Hermitian operators $\hat{A}$ and $\hat{B}$ on a quantum system in a mixed state. We introduce generalized measures of the mean-square deviations of $\hat{A}$ and $\hat{B}$ from their expectation values. These measures fully incorporate *correlations* in the density operator, which are specified by off-diagonal elements of the latter in the Hilbert space of the appropriate observables. We then utilize the generalized variances to derive the associated Heisenberg-type uncertainty relations, and we show that the UR's that we derive reduce to the conventional one for pure states. Next, we demonstrate that the explicit incorporation of quantum coherences into the structure of the UR's relaxes the requirement that the minimum-uncertainty states be pure. In particular, we find that in the important case of the coordinate and momentum operators, the most general minimum-uncertainty states are *displaced squeezed thermal* states. Further, we compare our algebraic approach to the study of correlations in a mixed state of a quantum system with the phase-space formalism employed in Ref. [17]. Finally, we note the similarity between a special case of the uncertainty relations and a recently obtained reciprocity inequality for partially coherent light treated classically [18].

## II. GENERALIZED VARIANCES AND ASSOCIATED UNCERTAINTY RELATIONS

We begin our analysis by introducing generalized variances of the observable $\hat{A}$ that (i) explicitly incorporate quantum coherences and (ii) reduce to the conventional measure of the mean-square deviation from the expectation value of $\hat{A}$ for the measured system in a pure state. Let

 

$$\langle(\Delta\hat{A})^2\rangle_+ \equiv \mathrm{Tr}([\Delta\hat{A},\hat{\rho}]_+^2), \tag{2a}$$

and

$$\langle(\Delta\hat{A})^2\rangle_- \equiv -\mathrm{Tr}([\Delta\hat{A},\hat{\rho}]_-^2). \tag{2b}$$

Here, $[,]_+$ stands for the anticommutator of two operators, $[\hat{A},\hat{B}]_+ \equiv \hat{A}\hat{B}+\hat{B}\hat{A}$, and $\Delta\hat{A}=\hat{A}-\langle A\rangle$, where $\langle A\rangle=\mathrm{Tr}(\hat{A}\hat{\rho})$ is the expectation value of the operator $\hat{A}$. It should be noted that the minus sign in front of the right-hand side of Eq. (2b) is necessary to ensure the nonnegativity of the variance. Further, one can express the right-hand sides of Eqs. (2a) and (2b) in terms of the eigenstates of $\hat{A}$, $\hat{A}|a\rangle=a|a\rangle$. By a straightforward calculation, one can show that if the system is in a pure state $|\psi\rangle$ both the generalized variances are equal to the usual variance up to a numerical factor: $\langle(\Delta\hat{A})^2\rangle_-^{(\text{pure})}=\langle(\Delta\hat{A})^2\rangle_+^{(\text{pure})}=2(\langle\psi|\hat{A}^2|\psi\rangle-\langle\psi|\hat{A}|\psi\rangle^2)$.

The two generalized variances have different physical meaning. In order to see this, we expand Eqs. (2a) and (2b) in the basis $\{|a\rangle\}$ of the eigenstates of $\hat{A}$, assuming, for simplicity, that the expectation value of $\hat{A}$ is zero. The variance defined by Eq. (2a) may then be expressed as [19]

$$\langle(\Delta\hat{A})^2\rangle_+ = \sum_{a,a'}(a+a')^2|\langle a|\rho|a'\rangle|^2. \tag{3}$$

It follows from this definition that the right-hand side is a generalized mean-square deviation from the expected value, taken to be zero, of the observable $\hat{A}$. The expression $|\langle a|\rho|a'\rangle|^2$ represents a generalized probability distribution, which is not restricted to the diagonal elements of $\hat{\rho}$ as it is for the variance. The other generalized variance, given by the expression

$$\langle(\Delta\hat{A})^2\rangle_- = \sum_{a,a'}(a-a')^2|\langle a|\rho|a'\rangle|^2, \tag{4}$$

characterizes the effective width of the correlations between any pair of eigenvalues of the observable $\hat{A}$ in the mixed state of the system. It is to be also noted that if the operator $\hat{A}$ is simultaneously measurable with $\hat{\rho}$, the density operator is diagonal in the eigenstates of $\hat{A}$, and the generalized variance, given by Eq. (4), identically vanishes. We will now state our main result pertaining to the intrinsic spreads of a pair of noncommuting observables in an open quantum system.

*Theorem. The generalized variances of any pair of noncommuting observables $\hat{A}$ and $\hat{B}$ measured separately on an open quantum system, which is described by the density operator $\hat{\rho}$, satisfy the pair of uncertainty relations*

$$\langle(\Delta\hat{A})^2\rangle_-\langle(\Delta\hat{B})^2\rangle_+ \geqslant |\mathrm{Tr}([\hat{A},\hat{B}]_-\hat{\rho}^2)|^2, \tag{5a}$$

$$\langle(\Delta\hat{B})^2\rangle_-\langle(\Delta\hat{A})^2\rangle_+ \geqslant |\mathrm{Tr}([\hat{A},\hat{B}]_-\hat{\rho}^2)|^2. \tag{5b}$$

*Proof.* It is sufficient to prove either of the two inequalities, say, (5a). Consider the auxiliary inequality

$$\mathrm{Tr}(\hat{F}^\dagger\hat{F}) \geqslant 0, \tag{6}$$

where $\hat{F}$ is an arbitrary operator. Suppose that

$$\hat{F}=[\Delta\hat{B},\hat{\rho}]_+ + i\lambda[\Delta\hat{A},\hat{\rho}]_-, \tag{7}$$

where $\lambda$ is any real number. We substitute for $\hat{F}$ from Eq. (7) into Eq. (6) and after some algebra, involving the invariance of a trace with respect to cyclic permutations of the operators, we arrive at the inequality

$$\mathrm{Tr}([\Delta\hat{B},\hat{\rho}]_+^2)-\lambda^2\mathrm{Tr}([\Delta\hat{A},\hat{\rho}]_-^2)-2i\lambda\,\mathrm{Tr}([\hat{A},\hat{B}]_-\hat{\rho}^2)\geqslant 0. \tag{8}$$

On making use of the definitions (2), it follows that

$$\lambda^2\langle(\Delta\hat{A})^2\rangle_- + \langle(\Delta\hat{B})^2\rangle_+ + 2\lambda|\mathrm{Tr}([\hat{A},\hat{B}]_-\hat{\rho}^2)| \geqslant 0. \tag{9}$$

This inequality holds regardless of the value of $\lambda$ provided that

$$|\mathrm{Tr}([\hat{A},\hat{B}]_-\hat{\rho}^2)|^2-\langle(\Delta\hat{A})^2\rangle_-\langle(\Delta\hat{B})^2\rangle_+ \leqslant 0, \tag{10}$$

which is equivalent to the inequality (5a) that we set out to prove. The other inequality, (5b), can be proven in a similar way.

In the limiting case when either of the two operators, say $\hat{A}$, commutes with $\hat{\rho}$, the left-hand side of the inequality (5a) identically vanishes. It follows at once from the properties of traces that in this limiting case, the right-hand side of inequality (5a) is also zero. Further, when the quantum system is in a pure state, the UR's that we have just established reduce to the usual Heisenberg-type uncertainty relation (1). However, the UR's (5) are quite different from the conventional generalized uncertainty relations [13] for the quantum system in a mixed state.

The minimum-uncertainty state of the new uncertainty relations corresponds to the case when the equality sign in inequality (6) holds, i.e., when $\mathrm{Tr}(\hat{F}^\dagger\hat{F})=0$. This equation implies that all the eigenvalues of $\hat{F}$ must be zero. In other words, $\hat{F}$ must be a null operator, i.e., $\hat{F}|\nu\rangle=0$, for any basis states $\{|\nu\rangle\}$. It then follows from Eq. (7) that

$$[\Delta\hat{B},\hat{\rho}]_+ + i\lambda[\Delta\hat{A},\hat{\rho}]_- = 0. \tag{11}$$

This equation, which determines the density operator of the most general states minimizing the new UR's for an *arbitrary* pair of noncommuting operators, is another key result of this paper. It can be concluded from Eq. (11) that the incorporation of correlations in the quantum state into the definition of uncertainties makes it possible for a *mixed state* to be a minimum-uncertainty state of the corresponding UR's.

### III. MINIMUM UNCERTAINTY STATES FOR THE COORDINATE-MOMENTUM PAIR: SQUEEZED THERMAL STATES

We will now apply the above general results to the important special case of the coordinate and momentum operators $\hat{x}$ and $\hat{p}$. In particular, we will determine the most general states of an open quantum system that minimize the uncertainties of the expectation values of $\hat{x}$ and $\hat{p}$. To this end, we rewrite the general equation (11) for the coordinate and momentum operators assuming, to begin with, that their expectation values in the minimum-uncertainty state are zero, i.e., that $\langle x\rangle=0$ and $\langle p\rangle=0$. We then have

$$[\hat{x},\hat{\rho}]_{+}+i\mu[\hat{p},\hat{\rho}]_{-}=0, \tag{12}$$

where $\mu$ is a real number. Next, we convert Eq. (12) to the coordinate representation. The resulting equation is

$$\left(\frac{\partial}{\partial x'}+\frac{\partial}{\partial x}+\frac{x+x'}{\mu}\right)\langle x'|\rho|x\rangle=0. \tag{13}$$

A general solution to Eq. (13) subject to the hermiticity condition $\langle x'|\rho|x\rangle=\overline{\langle x|\rho|x'\rangle}$ is

$$\langle x'|\rho|x\rangle=A\exp\left[-\frac{(x^2+x'^2)}{2\delta_1^2}+\frac{xx'}{\delta_2^2}\right], \tag{14}$$

where $A$ is a normalization constant, both $\delta_1$ and $\delta_2$ are characteristic spatial scales. Further, the positive definiteness of the density matrix imposes a constraint on the values of $\delta_1$ and $\delta_2$, namely, $\delta_2>\delta_1$. On substituting for $\langle x'|\rho|x\rangle$ from Eq. (14) into Eq. (13), one obtains for $\mu$ the expression $1/\mu=1/\delta_1^2-1/\delta_2^2$.

We will now show that the density operator, given by the matrix elements (14), corresponds to a thermal state. For this purpose, we recall that the density operator of the thermal state in the number representation is

$$\hat{\rho}_0=\frac{1}{Z}\sum_n e^{-\beta\omega n}|n\rangle\langle n|. \tag{15}$$

Here, $Z=1-e^{-\beta\omega}$ is a normalization constant, $\omega$ is an oscillator frequency, $\beta$ is the inverse temperature, where we use units such that the Planck constant $\hbar$ and the Boltzmann constant are both equal to unity. One can then determine the matrix elements of the density operator (15) in the coordinate representation by making use of the explicit expressions for the oscillator wave functions in this representation viz.,

$$\langle x|n\rangle=\left(\frac{1}{2^n n!\,l_0\sqrt{\pi}}\right)^{1/2}\exp(-x^2/2l_0^2)H_n(x/l_0), \tag{16}$$

and by employing Mehler's formula [20]

$$\frac{1}{\sqrt{1-z^2}}\exp\left[\frac{2xy-(x^2+y^2)z}{1/z-z}\right]=\sum_{n=0}^{\infty}\frac{(z/2)^n}{n!}H_n(x)H_n(y). \tag{17}$$

In Eq. (16), $l_0$ is a characteristic width of the ground-state wave function of the oscillator with the frequency $\omega$, and $H_n(x)$ is the Hermite polynomial of order $n$. The resulting expression for the matrix elements of the density operator of the thermal state, in the coordinate representation, is

$$\langle x'|\rho_0|x\rangle=B\exp\left[-\frac{(x^2+x'^2)}{2l_0^2}\coth(\beta\omega)\right]\times\exp\left(\frac{xx'}{l_0^2\sinh(\beta\omega)}\right), \tag{18}$$

where $B$ is a normalization constant. On comparing Eq. (18) with Eq. (14), we conclude that the thermal state with $\delta_1^2=l_0^2\tanh(\beta\omega)$ and $\delta_2^2=l_0^2\sinh(\beta\omega)$ indeed minimizes the UR's in the case when $\hat{A}=\hat{x}$ and $\hat{B}=\hat{p}$.

Actually, the thermal state is not the only minimum-uncertainty state for the coordinate-momentum pair. We will now demonstrate that all the minimum-uncertainty states are related by a unitary transformation. To this end, we rewrite the equation for the density operator of the minimizing states, satisfying Eq. (12), in the form

$$\left(\frac{\hat{x}}{\sqrt{\mu}}+i\sqrt{\mu}\hat{p}\right)\hat{\rho}_0+\hat{\rho}_0\left(\frac{\hat{x}}{\sqrt{\mu}}-i\sqrt{\mu}\hat{p}\right)=0. \tag{19}$$

Next, we introduce the scaled creation and annihilation operators by the expressions

$$\hat{a}=\frac{1}{\sqrt{2}}\left(\lambda\hat{x}+\frac{i}{\lambda}\hat{p}\right), \tag{20a}$$

$$\hat{a}^{\dagger}=\frac{1}{\sqrt{2}}\left(\lambda\hat{x}-\frac{i}{\lambda}\hat{p}\right), \tag{20b}$$

where $\lambda$ is a real positive scaling factor. On substituting for $\hat{x}$ and $\hat{p}$ from Eqs. (20) into Eq. (19), we obtain the equation

$$(\hat{a}\cosh r+\hat{a}^{\dagger}\sinh r)\hat{\rho}_0+\hat{\rho}_0(\hat{a}\sinh r+\hat{a}^{\dagger}\cosh r)=0, \tag{21}$$

where

$$\sinh r=\frac{1}{2}\left(\frac{1}{\lambda\sqrt{\mu}}-\lambda\sqrt{\mu}\right), \tag{22a}$$

and

$$\cosh r=\frac{1}{2}\left(\frac{1}{\lambda\sqrt{\mu}}+\lambda\sqrt{\mu}\right). \tag{22b}$$

It follows at once from Eq. (19), Eq. (21), and Eqs. (22), that if the scaling parameter $\lambda=1/\sqrt{\mu}$, then the thermal state solution to Eq. (19) is recovered. In order to find a general class of minimizing states, one can identify, following the method developed in Ref. [21] for the pure minimum-uncertainty states of the conventional UR, the unitary operator

$$\hat{S}=\exp\left[\frac{1}{2}(z\hat{a}^2-z^*\hat{a}^{\dagger 2})\right], \tag{23}$$

which performs the squeezing transformation:

$$\hat{S}\hat{a}\hat{S}^\dagger=\hat{a}\cosh r+\hat{a}^\dagger e^{-i\theta}\sinh r. \tag{24}$$

Here, the complex squeeze parameter $z$ is given by $z=re^{i\theta}$. On comparing the right-hand side of Eq. (24) with the left-hand side of Eq. (21), we conclude that the expressions in the brackets in Eq. (21) can be obtained by the action of $\hat{S}$ with a real squeeze parameter, $(\theta=0)$, onto the creation and annihilation operators. Equivalently, one can apply a unitary squeezing transformation to the thermal density operator,

$$\hat{\rho}=\hat{S}^\dagger\hat{\rho}_0\hat{S}. \tag{25}$$

Equation (25) represents a general state which minimizes the UR's (5), in the special case when $\hat{A}=\hat{x}$ and $\hat{B}=\hat{p}$.

To account for the statistical properties of the state given by the density operator (25), we calculate the Husimi $Q$ function of such a state, defined as

$$Q(\alpha)=\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \tag{26}$$

where $|\alpha\rangle$ is a coherent state. Starting with the density operator (15) of the thermal state, we apply the squeezing transformation $\hat{S}$ and take diagonal elements of the resulting density operator in the coherent-state representation with the help of the expression [22],

$$\langle n|r,\alpha\rangle=\frac{t^n}{\sqrt{c_r n!}}\exp\left(-\frac{1}{2}|\alpha|^2+\alpha^2t^2\right)H_n\left(\frac{\alpha}{2c_rt}\right). \tag{27}$$

Here $|r,\alpha\rangle=\hat{S}|\alpha\rangle$ is a pure squeezed state, and $t=(s_r/2c_r)^{1/2}$, where $c_r\equiv\cosh r$ and $s_r\equiv\sinh r$. Next, performing the summation with respect to $n$ with the aid of Mehler's formula (17), we find that

$$Q(x_1,x_2)=\frac{\operatorname{sech} r}{\pi(1-\zeta)\sqrt{1-\zeta^2\tanh^2 r}}\exp\left(-\sum_{j=1,2}\frac{x_j^2}{2\sigma_j^2}\right), \tag{28}$$

where the quadrature variables are defined as $x_1=\alpha+\alpha^*$ and $x_2=i(\alpha^*-\alpha)$. Further, $\zeta=e^{-\beta\omega}$, and the quadrature variances are found to be given by the expressions

$$\sigma_1=\frac{1}{1+e^{2r}}-\frac{\zeta\operatorname{sech}^2 r}{2(1+\zeta\tanh r)}, \tag{29a}$$

$$\sigma_2=\frac{1}{1+e^{-2r}}-\frac{\zeta\operatorname{sech}^2 r}{2(1-\zeta\tanh r)}. \tag{29b}$$

The minimum-uncertainty state with the $Q$ function represented by Eq. (28) is well known in quantum optics as a squeezed thermal state. It can be produced, for instance, by degenerate four-wave mixing in a cavity coupled to a thermal reservoir in order to model relaxation processes [23]. It is to be also noted that there is a continuous parameter, $\zeta$, which specifies the degree of decorrelation of the state due to the interaction of the system with the environment. It is seen from Eq. (29) that in the limiting case as $\zeta\to 0$, which corresponds to a zero temperature of the environment, the state is an ideal squeezed vacuum. Our results for the minimum-uncertainty state can be readily generalized to the case of arbitrary linear combinations of coordinate and momentum, which may represent, for example, the quadratures of the electromagnetic field in the context of quantum optics. The analysis indicates that in this case, the minimum-uncertainty state is a displaced squeezed thermal state, with the density operator

$$\hat{\rho}=\hat{S}^\dagger\hat{D}^\dagger\hat{\rho}_0\hat{D}\hat{S}. \tag{30}$$

Here, $\hat{D}$ is the displacement operator that shifts the thermal state by an amount equal to a nonzero expectation value of a particular quadrature.

## IV. DISCUSSION AND SUMMARY

It is instructive to compare our results with some previous attempts to generalize the uncertainty relation to open quantum systems. To our knowledge, the only investigation that incorporates correlations in a quantum state into the structure of the UR is that of Chountasis and Vourdas [17]. These authors introduced variances of the coordinate and momentum in an extended phase space and applied a Fourier-transform relationship between the Wigner and the Weyl distributions, to derive the corresponding uncertainty relations. It was also verified in Ref. [17] that the thermal state minimizes such UR's. In the special case of the coordinate-momentum pair, our UR's reduce to the uncertainty relations derived in Ref. [17], apart from a normalization factor. However, our approach is not restricted to the coordinate-momentum pair, being applicable to any pair of arbitrary noncommuting operators $\hat{A}$ and $\hat{B}$. Moreover, the general algebraic approach that we have developed makes it possible to define squeezing for open quantum systems, and hence, to identify the most general minimum-uncertainty states for the coordinate-momentum pair, which turn out to be the squeezed thermal states. We mention that there exists a formal analogy between quantum mechanics of open systems and classical optics of partially coherent light. The density operator describing a mixed state is, in a sense, analogous to the cross-spectral density of such light. Moreover, at least at a qualitative level, the counterpart in classical optics of the degree of correlations in the mixed state of a quantum system is the degree of spatial coherence of light, which is also known as the spectral degree of coherence (Ref. [22], Chap. 4). The question of the appropriate reciprocity relations, analogous to the quantum mechanical UR's, for partially coherent light has already been addressed (see Refs.

[18,24,25]). In this context, the recently obtained reciprocity relation between the effective coherence size of a partially coherent source and the width of the angular correlations in the beam generated by such a source [18] is similar to the quantum mechanical UR's (5), specialized to the coordinate-momentum pair. However, the possibility of squeezing of the variance of either of the two conjugate variables in the quantum system has no analogy in classical optics (Ref. [22], Chap. 21).

To summarize, we have studied the influence of correlations in a quantum state of an open system on the intrinsic uncertainties of the values of an arbitrary pair of noncommuting observables. The introduction of generalized variances, which take such correlations fully into account, together with the algebraic approach, which we developed, enabled us to derive Heisenberg-type uncertainty relations for such variances. The UR's, along with the usual uncertainty relations generalized to quantum systems in mixed states [13] impose, in general, different constraints on the spread of the values of conjugate observables, which are due to the fundamental indeterminacy of such a state. The possibility that there is a connection between the two sets of uncertainty relations for mixed states remains to be examined.

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